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Lecture Notes in Mathematics
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Computing Singularities by Gröbner Bases

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Preface

"How can anything a computer produces have to do with chaos? I thought computers were based on *logic*."

Inspector Morse, in response to an explanation of the Mandelbrot fractal.

This book deals with nonlinear Hamiltonian systems, depending on parameters. Such systems occur for example in the modeling of frictionless mechanics and optics. The general goal is to understand their dynamics in a qualitative, and if possible, also quantitative way. The dynamical behavior generally is expressed in terms of equilibria, periodic and quasi periodic solutions as well as corresponding homo– and heteroclinic connections between those. Such connections often are accompanied by chaotic dynamics. In many important cases, it is possible to reduce a skeleton of the dynamics to lower dimensions, sometimes leading to a Hamiltonian system in one degree of freedom. Such reduced systems allow a singularity theory or catastrophe theory approach which gives rise to transparent, in a sense polynomial, normal forms. Moreover the whole process of arriving at these normal forms is algorithmic. The purpose of this book is to develop computer-algebraic tools for the implementation of these algorithms, which involves Gröbner basis techniques. This set-up allows for many applications concerning resonances in coupled or driven oscillators, the n-body problem, the dynamics of the rigid body, etc. in Hamiltonian mechanics. A version of the spring–pendulum is used as a test case.

The present work appeared earlier as the PhD thesis [Lun99a] of Gerton Lunter, written under supervision of Henk Broer and Gert Vegter.

Background

General background of this work is the theory of nonlinear Dynamical Systems as it has evolved since Poincaré [Poi87]. A main aspect of this development is that the 19th century programme of the explicit analytic computation of individual evolutions (solutions) largely has broken down. Indeed, since then the emphasis has shifted to considering the whole phase space as it contains all possible evolutions, in simple cases expressed by devices like phase portraits.

More generally, increasing importance was given to the study of geometric objects like tori and stable or unstable manifolds, and their relative positions in phase space with respect to each other, periodic evolutions, and equilibria.

Another development was that the interest moved away from individual systems, and towards the study of dynamical properties that are persistent for small perturbations of the given system, compare the treatment of the term 'genericity' by Smale [Sma67] in the 1960s. Parallel to this development is the introduction of parameters into dynamical systems, so as to effect deformations or unfoldings of these in a systematic way, compare Thom [Tho72], also see Arnold [Arn88].

Summarizing one could say that during one century, the emphasis in dynamical systems shifted from the explicit analytic computation of a given evolution to the consideration of generic properties of families of dynamical systems deforming a given one, where the methods became more geometric and qualitative.

In a parallel development, during the past 50 years the electronic computer became increasingly important for studying concrete dynamical systems, possibly depending on parameters. On the one hand we saw visualization of phase portraits, chaos, invariant manifolds, bifurcation diagrams, etc., while on the other hand also the computation of underlying dynamical characteristics like normal form coefficients, dimensions, Lyapunov exponents, power spectra, etc., became possible. Here, apart from purely numerical computations, also symbolic computations and computer algebra plays a role.

The present book is focused on various forms of normal form computations, which are deeply involved with computer algebra. We restrict to the Hamiltonian context, where moreover the systems can be approximated by those admitting a reduction to one degree of freedom. Therefore, apart from the approximation aspect, we are back in the paradise of the 19th century. The one degree of freedom reduction supports the coherent dynamics of the original model, while it also provides a skeleton for the chaos.

By this combination of geometric and algebraic methods, the quantitative element resurfaces: the geometric descriptions can be traced in detail to the original physical model equations.

Formal normal forms, a perturbation problem

Several methods exist by which dynamical systems can be reduced to lower dimensions. One standard way is by restricting the system to an invariant manifold, like a center manifold. Factoring out symmetry is another way to reduce the phase space. In the Hamiltonian context this also is the classical approach, since symmetry by Noether's theorem [Arn89] is related to the existence of (first) integrals and therefore conservation laws. A textbook example of this is the Kepler problem, where rotational symmetry thus leads to the conservation of angular momentum and a cyclic variable, which allows reduction to one degree of freedom. In this example the symmetry group is the circle or a 1-torus. In many applications the symmetry group is a higher dimensional torus.

In the present Hamiltonian context we concentrate on reduction by symmetry. Within the class of general Hamiltonian systems, those that admit a symmetry group like an n-torus, are highly degenerate: in other words, having such a torus symmetry is not a generic property. Nevertheless, the theory of formal normal forms near certain equilibria or (quasi–) periodic orbits admits the local approximation of the given system by a symmetric one. This method of simplifying formal series goes back to Poincaré and Birkhoff [Poi28, Bir50] also see Gustafson [Gus66]. For general information, see e.g. Arnold [Arn89, Arn88].

To be somewhat more precise, if the original vector field is denoted by X, where for the moment parameter dependence is suppressed, then the present normal form theory asks for a canonical transformation Φ, such that

$$\Phi_*(X) = N + R.$$

Here N is the normalized, symmetric part, that describes the slow dynamics after factoring out the symmetry, while R is the small remainder term. In this way the study of the dynamical system is divided into two parts. The first is to understand the symmetric approximation N and the second to take the perturbation R into account. Since this book will be entirely devoted to the former of these two problems, we just give a few remarks about the latter, e.g., see Broer and Takens [BT89]. Indeed, in this setting the occurrence of separatrix splitting as associated to chaos, is a flat phenomenon. One could say that the approximation N contains the regular skeleton supporting the chaotic zones of instability.

Remarks

- The expression $\Phi_*(X) = N+R$ is reminiscent to the division algorithm. Below this fact will be elaborated further in the context of Gröbner basis techniques [CLO92].
- Concerning regularity, our context mainly is assumed to be C^∞. However, if one restricts to the real analytic C^ω case, the remainder term R can be estimated in an appropriate exponential way. Compare Neishtadt [Nei84, BRS96, BR01].

Singularity and Catastrophe Theory, polynomial normal forms

In many examples factoring out the symmetry leads to Hamiltonian systems in one degree of freedom where the dynamics largely is determined by the level curves of the corresponding Hamilton functions. The presence of parameters in the original problem so brings us in the setting of families of real functions in dimension 2. Since the reduced phase space is 2-dimensional, in a further simplification process we may abstain from symplecticity of the transformations, since this only affects the time parametrization of the dynamics and not the level curves of the functions. Now singularity and catastrophe theory normal forms

can be computed, which to a large extent are polynomial. For a general reference, e.g., see Thom [Tho72], Bröcker and Lander [BL75].

Regarding the reduction to one degree of freedom Hamiltonian systems, we have selected two different approaches. The first of these is the *planar reduction method*, compare Broer and Vegter *et al.* [BV92, BCKV93, BCKV95], comparable to the classical Keplerian reduction. It is to be noted that the formal integrals obtained after the formal normalization show up as distinguished parameters in the singularity theory. The second approach concerns the *energy–momentum map*, compare Duistermaat [Dui84], Van der Meer [Mee85], Cushman and Bates [CB97]. Also see [BHLV98]. Both methods, after a formal normal form, reduce to one degree of freedom Hamiltonian systems after which singularity theory is used. As said before, here further simplifying transformation are applied. In the former case this leads to so-called *right equivalences* and in the latter to *left–right equivalences*.

The most interesting cases contain rather strong resonances, which also gives discrete symmetries in the normal form. Furthermore, certain discrete symmetries are considered that are *a priori* to the original physical problem, such as time reversibility.

For general background on the use of singularity and catastrophe theory to dynamical systems, frequently using contact equivalence, see Golubitsky, Schaeffer, Stewart and Marsden [GS85, GSS88, GMSD95]. In these references, as well as in Wassermann [Was75], also distinguished parameters play a role. However, in view of the special nature of the distinguished parameters, which are nonnegative action variables, a new unfolding theory was developed in [BCKV93]. For general background also see Damon [Dam84, Dam88, Dam95] and Montaldi [Mon91].

Algorithms, setting of the problem

An elementary observation is that, without the help of computers, the computations mentioned so far can not be extended to the level that is of interest for serious applications. A good example of this in the dissipative setting is given by Marsden and McCracken [MM76]. Fortunately the proofs involved here are highly constructive and lead to algorithms that can easily be implemented on computers.

The algorithm of formal normal forms already has been widely implemented on computers. In the Netherlands e.g. the Dynamical Systems Laboratory at the CWI Amsterdam has been active in this, unifying the process and making it more sophisticated. The singularity theory normal form also is algorithmic in nature. See Kas and Schlessinger [KS72]. Here, however, implementation largely was lacking, while the complication of practical computations renders the use of computers essential. As indicated before, this program is deeply involved with computer algebra, in particular with Gröbner basis techniques [CLO92]. It turns out that the methods of planar reduction and the energy–momentum map may be formalized in a unified way with help of *standard bases*. A recent reference in

this direction of applying computational algebra to dynamical systems is Gatermann [Gat00].

One of the key ideas is to keep track of all normalizing transformations, which makes it possible to translate the mathematical conclusions back into the original physical context of the model. This task is effectively carried out for a mechanical example. In that sense, key aspects of the qualitative model are made quantitative. For a general reference on the application of computer algebraic methods in perturbation theory see Rand and Armbruster [RA87].

The scope and beyond

Extension of the research at hand can be pursued in the following directions. As introductory reduction algorithms also the Liapunov–Schmidt or the Moser–Weinstein method, as well as the restriction to center manifolds, may be taken into consideration. After this again a singularity and catastrophe theory approach seems feasible. Again compare [GMSD95]. Moreover, combinations of these approaches are of importance.

Another option is to incorporate the work of Hummel [Hum79], who investigates periodic points of diffeomorphisms by contact equivalence. Also here discrete symmetries are essential. For earlier, theoretical results compare Takens [Tak74a] and see [GS85, GSS88, GMSD95, BGV02].

Also at the level of concrete applications and case studies many extensions are possible, beyond the present case study of a spring pendulum and the resonances considered here. We just mention mechanical examples like driven or coupled oscillators (compare, e.g., [TRVN00]), the rigid body, etc. For earlier work in this direction compare, [Han95a, Han95b, Hov92, BHvN98, BHvNV99]. Also see [Lun99a, Lun99b]. Moreover, the research at hand forms the beginning of a theoretical basis for the future development of a coherent set (package or library) of computer programs, suitable for further use.

A further development that may be of great importance is the combination of computer algebraic and numerical means, compare Simó [Sim89]. In this way it becomes possible to be more efficient in dealing with the constants, parameters and transformations, only keeping track of essential things. Another aspect is the computation and visualization of invariant manifolds, e.g., compare Vegter *et al.*, [HOV95, BOV97].

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Table of Contents

1 Introduction

Our concern is with Hamiltonian systems that by symmetry are reducible to one degree of freedom (or at least reducible to a large approximation). As a concrete example and as a case study for our approach we often consider a model for the spring–pendulum near one of its stronger resonances. For motivation and background the reader is referred to our extensive preface.

Focusing down further, we shall formulate the main goal of the present book. After reduction of the (approximating) Hamiltonian system by symmetry, we apply singularity theory to obtain transparent normal forms for the dynamical skeleton. For this we need coordinate transformations and reparametrizations, all of which can be obtained in an algorithmic way. It is our purpose to develop computer algebraic methods for this. Therefore, the emphasis lies on the *algorithmic methods*, especially in later chapters.

The earlier chapters are dealing with the translation between the context of dynamical systems on the one hand and that of the computer algebraic implementation of the singularity theory algorithms on the other hand.

Two reduction methods have been selected. These methods, dubbed the *planar reduction* [BCKV93] and *energy–momentum map* [Cus83, Dui84, Mee85] methods, both apply Birkhoff normal form transformation to obtain an (approximate) system with symmetry, and then proceed with reduction to a one degree of freedom system, in different ways. Algorithms for computing the reparametrizations involved are developed in chapter 4 onwards, and find application in both reduction methods in chapters 2 and 3. The algorithmic approach also enabled us to compute nondegeneracy conditions. Some of these were already found in [Dui84], also see [GMSD95], whereas certain others are rather hard to find by paper-and-pencil calculations.

As a byproduct of developing these algorithms we gain a deeper insight in the two reduction methods. Both methods are applied (and tested) on the concrete example of the spring–pendulum in a few strong resonances close to equilibrium. This enables us to make comparisons between the two methods, concerning their applicability and the strength of their dynamical conclusions. See Sect. 1.2 for a discussion.

Partly summarizing, we mention that the key ideas for making the planar reduction method algorithmic, and hence constructive, with suitable changes, are also applicable to the energy–momentum map method. In fact, this led to a

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unifying approach to both methods, presented in chapters 6 and 7. The relevant notion for this unification is the standard basis, which deepened the understanding of the algorithms, and facilitated their derivation. Indeed, standard bases, and the Gröbner bases of which they are a generalization, prove to fit several ideas from both methods in a rather unexpected way.

1.1 A further setting of the problem

In this section, we explain the two reduction methods considered, by applying them to a simple two degree-of-freedom system. Our goal of explicitly computing coordinate transformations and reparametrizations, boils down to solving the so-called *infinitesimal stability equation*. In the context of polynomial rings, an efficient procedure based on Gröbner bases solves this equation. The two reduction methods motivate two different generalizations of this idea, leading *standard bases*.

1.1.1 The planar reduction method

The starting point is an article by Broer, Chow, Kim and Vegter [BCKV93], in which a two-step reduction method is used to find a polynomial model for a certain class of Hamiltonian dynamical systems. The reduction starts with a Birkhoff normalization. The resulting near $\mathbb{S}^1$ symmetry then is used to reduce to a planar one-degree-of-freedom system. Subsequently, right-equivalences (i.e., planar coordinate changes that are not necessarily symplectic) are used to find a polynomial and versal model system. Both steps are qualitative, in the sense explained above. On the one hand, to pull quantitative information through the Birkhoff normalization is feasible, as it was long known how to compute the associated (symplectic) conjugacy explicitly. On the other hand, for the versal deformation such explicit computations were, to our knowledge, not done before.

As said earlier, our aim is to see how much quantitative information about the original dynamical system could be mustered by computing the conjugacies along the lines of the planar reduction method. This program was presented in [BLV98] and carried out in detail in [BHLV98].

We now illustrate the planar reduction method by a simple example; for more details see [BCKV93, BCKV95]. Suppose we have a Hamiltonian living on $\mathbb{R}^4$ with a degenerate quadratic part, whose Hessian has corank 1. Then after Birkhoff normalization, truncation, and suitable time-reparametrizations, generically the following normal form results:

$$H(x_1, y_1, x_2, y_2) = \frac{1}{2}\left(x_1^2 + y_1^2\right) + \frac{1}{2}y_2^2 + \frac{1}{3}x_2^3 + g\left(\tfrac{1}{2}(x_1^2 + y_1^2), x_2, y_2\right), \tag{1.1}$$

with g containing all terms of degree 4 and higher. This system has an $\mathbb{S}^1$-symmetry (which is exact due to the truncation): rotation in the x_1, y_1-plane.

The semi-simple quadratic part $\lambda := \frac{1}{2}(x_1^2 + y_1^2)$ is the conserved quantity associated to this symmetry. We may treat λ as a parameter, and reduce to a planar system by ignoring the dynamics in x_1, y_1. This step is analogous to the reduction step in the Kepler problem. The resulting reduced system is

$$H^r(x_2, y_2, \lambda) = \frac{1}{2}y_2^2 + \frac{1}{3}x_2^3 + g(\lambda, x_2, y_2).$$

We now apply (non-symplectic) right transformations to normalize the system further. Since the system has one degree of freedom now, these transformations differ from symplectic ones by just a time reparametrization. The function that results when putting $\lambda = 0$ is called the *central singularity*. It may be brought into polynomial normal form $\frac{1}{2}y_2^2 + \frac{1}{3}x_2^3$ using the Splitting Lemma [BL75, Gib79, Mar82] and a classification theorem for singularities [Mar82, Sie74]. Hence we may assume that $g(0, x_2, y_2) \equiv 0$, that is, the system is now reduced to some deformation of the polynomial normal form.

The last step in the simplification process is finding a versal unfolding. To do this, we look at the tangent space to the orbit associated to the action of the group of right transformations on the central singularity. Locally, this tangent space may be identified with an ideal in the ring $\mathcal{E}_{x_2,y_2}$ of (germs of) functions depending on x_2, y_2. Generators of this ideal are $\partial/\partial x_2$ and $\partial/\partial y_2$ acting on the central singularity, resulting in the ideal $\langle x_2^2, y_2 \rangle$. As a real vector space, this is a codimension-1 subspace of the maximal ideal of $\mathcal{E}_{x_2,y_2}$ (also called the ring of (germs of) *potential functions*), and is complemented by, e.g., the 1-dimensional vector space spanned by the monomial x_2. The monomials chosen to complement the tangent space are called *deformation directions*. Adding the term μx_2 to the central singularity yields the versal unfolding:

$$H^u(x_2, y_2, \mu) = \frac{1}{2}y_2^2 + \frac{1}{3}x_2^3 + \mu x_2.$$

This happens to be the normal form of the *fold* catastrophe, one of the elementary catastrophes classified by René Thom [Tho72]. By the general theory it is known that H^r is conjugate to H^u by a reparametrization $\mu = \mu(\lambda)$, and a coordinate transformation (right-equivalence) ϕ of the x_2, y_2 depending on the parameter λ:

$$H^r(x_2, y_2, \lambda) = H^u(\phi(x_2, y_2, \lambda), \mu(\lambda));$$

we say that H^r is *induced* from H^u by ϕ and μ.

Up to this point we treated λ as an ordinary parameter, but actually it is a *distinguished* parameter, in the sense that it is a function on the 4-dimensional phase space. To reflect the special nature of λ in the normal form we restrict the class of allowed reparametrizations, so that changes in other parameters (which do not appear in our simple example) do not depend on λ, just as they do not depend on the phase variables x_2, y_2. Moreover, reparametrizations of λ are required to respect the zero level, which is natural since the 'action' λ is always positive. Following this idea (see also [BCKV93, BCKV95] and chapter 2), the final so-called *BCKV normal form* becomes

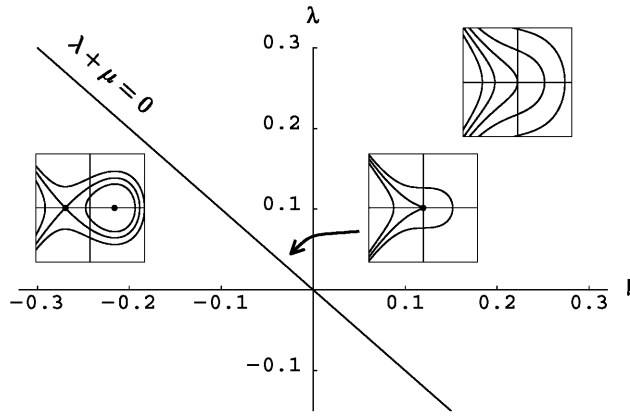

Fig. 1.1 Bifurcation diagram for (1.2): Hamiltonian saddle–node bifurcation

$$H^B(x_2, y_2, \lambda, \mu) = \frac{1}{2}y_2^2 + \frac{1}{3}x_2^3 + (\mu + \lambda)x_2. \tag{1.2}$$

The bifurcation diagram is depicted in Fig. 1.1. We now return to the original question: How to compute this reparametrization and coordinate transformation? An algorithm due to Kas and Schlessinger [KS72] details how to compute such conjugacies and essentially reduces the problem to solving several instances of the so-called *infinitesimal stability equation*, which is related to the tangent space and the versal unfolding. For our example, this equation takes the form

$$g(x_2, y_2) = \alpha_1(x_2, y_2)\frac{\partial F}{\partial x_2} + \alpha_2(x_2, y_2)\frac{\partial F}{\partial y_2} + \beta_1 x_2. \tag{1.3}$$

Here $F = \frac{1}{3}x_2^3 + \frac{1}{2}y_2^2$ is the central singularity; the expressions entering the infinitesimal stability equation are just the generators of the tangent space. The functions g are related to the $g(\lambda, x_2, y_2)$ above (in fact, they are the coefficients of $g(\lambda, x_2, y_2)$ expanded in powers of λ and the other parameters) and our task is to solve for functions α_i and real numbers β_i.

If the number of deformation parameters β_i is minimal, the β_i are uniquely determined, but in general the α_i are not. This is related to the fact that, for transformations inducing some deformation from a (uni)versal one, the *reparametrizations* are unique but the *coordinate transformations* generally are not. Algorithmically this is no problem – underdetermined systems are no more difficult to solve than uniquely determined ones.

Since we only want a finite part of the formal power series of the reparametrization and/or the coordinate transformation, solving the infinitesimal stability equation (1.3) is essentially a problem of finite-dimensional linear algebra. Though this solves our problem, two difficulties immediately emerge, one practical and another more fundamental in nature. On the practical side, the dimension of the matrices involved quickly becomes an issue: the number of monomials of

degree at most k increases rapidly with k. Since all matrix entries are symbolic expressions, solving the matrix equation soon becomes intractable – and many of such equations have to be solved.

More profound is the problem that the codimension of f, i.e., the number of deformation directions needed to complement the tangent space, can not be deduced straightforwardly from generators of this space. In other words, given generators of a formal power series ideal, it is in general not straightforward to find the highest order terms that determine its codimension as a real vector subspace of the ring.[1]

In the case of the *polynomial* ring, both problems are solved by Gröbner bases. Given arbitrary ideal generators, Buchberger's algorithm ([Buc65, BW93, CLO92], Sect. 6.4.1) yields generators of the same ideal that satisfy certain additional properties. These properties make computation of the codimension trivial, and can moreover be used to implement an efficient procedure for solving the infinitesimal stability equation (1.3) in the ring of polynomials. In [BHLV98] we developed methods suitable for formal power series rings, and applied it to the planar reduction method. We used the term *singularity Gröbner bases*, gleaned from [CW89]. Later it turned out that very similar ideas had already been developed, e.g., in [Mor82], using ideas of Hironaka (see [Hir64]). Hironaka coined the term *standard bases* for the analogue of Gröbner bases in formal power series rings. (Actually, Hironaka and Gröbner worked independently, with Hironaka's work preceding that of Gröbner by one year.)

1.1.2 The energy–momentum map

Next, we turn to the energy–momentum map reduction method, using the same example as before and also using the same terminology. The idea is to normalize a map from phase space to $\mathbb{R}^2$, whose two components are the Hamiltonian, and the conserved quantity λ associated to the $\mathbb{S}^1$-symmetry. (In this context, this quantity is referred to as the *momentum*.) This may be contrasted to the planar reduction method, which treats the Hamiltonian as the central object, and λ as a distinguished parameter. Since both the Hamiltonian and λ are conserved, fibers of the energy–momentum map are dynamically invariant.

In our example the system has two degrees of freedom. Then, after dividing out the symmetry, the fibers are 1 dimensional, so that they correspond to *orbits* of the system (though without time-parametrization). Since the set of fibers smoothly deforms when the energy–momentum map is subjected to smooth left–right transformations, it is natural to use these equivalences to bring the map in normal form.

Assume the system is already in (truncated) Birkhoff normal form (1.1), so that it has an exact $\mathbb{S}^1$-symmetry. We divide out this symmetry by going to

[1] However, it should be noted that in our particular example, the computation of the codimension can be easily handled by *ad hoc* methods. This is not true anymore for the reduction method that uses the energy–momentum map.

invariant coordinates. In this case $\rho_1 := \frac{1}{2}(x_1^2 + y_1^2)$, x_2 and y_2 together form a Hilbert basis (minimal set of generators; see [CLO92, Gat00, Hil93]) of the space of $\mathbb{S}^1$-invariant functions. In these coordinates, the energy–momentum map $\boldsymbol{E} : \mathbb{R}^3 \to \mathbb{R}^2$ takes the form

$$\boldsymbol{E} : (\rho_1, x_2, y_2) \mapsto (\ H\ ,\ \lambda\ ) = \left(\ \rho_1 + \frac{1}{2}y_2^2 + \frac{1}{3}x_2^3 + g(\rho_1, x_2, y_2)\ ,\ \rho_1\ \right).$$

In order to compute $\boldsymbol{E}$'s normal form, we have to know the tangent space of its orbit under left–right equivalences (B, A). These equivalences form a *group*, with operation $(B', A') * (B, A) = (B' \circ B, A \circ A')$, and deform $\boldsymbol{E}$ in the following way:

$$(B, A) : \quad \boldsymbol{E} \mapsto B \circ \boldsymbol{E} \circ A,$$

where B deforms the range of $\boldsymbol{E}$, and A is $\mathbb{S}^1$-equivariant. Generators of $\mathbb{S}_1$-equivariant vector fields (on $\mathbb{R}^4$) are $\frac{1}{2}(x_1\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_1})$, $y_1\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial y_1}$, $\frac{\partial}{\partial x_2}$ and $\frac{\partial}{\partial y_2}$; in invariant coordinates the first becomes $\rho_1\frac{\partial}{\partial \rho_1}$ whereas the second vanishes. (It follows that, in invariant coordinates, $\mathbb{S}^1$-equivariance of A boils down to the requirement that the subspace $\rho_1 = 0$ be invariant.) We want to compute the tangent space to the orbit of $\boldsymbol{E}$ under such transformations. It turns out that this tangent space is not an ideal, as it was in the previous example. This makes computing the codimension technically difficult; see, e.g., the remark in [Mar82, p. 183]. These difficulties only become more profound when explicit reparametrizations are computed. In order to show what form the tangent space takes, we need to go into some detail here.

Working with maps is less convenient than working with functions, so we try to reduce to the latter case. A small computation shows that the projection of the tangent space onto the second component is surjective; hence, we may restrict to left–right transformations that leave $\boldsymbol{E}$'s second component invariant, and compute the tangent space of the orbit of $\boldsymbol{E}$'s *first* component under such transformations. (See Sect. 3.2.3 for an elaboration of this argument.) We call this the *reduced tangent space*.

Let us denote coordinates on the range of $\boldsymbol{E}$ by ζ_1, ζ_2. Tangent vectors to the space of left–right transformations (B, A) are pairs of vector fields, on $\mathbb{R}^2 \ni (\zeta_1, \zeta_2)$ and $\mathbb{R}^3 \ni (\rho_1, x_1, x_2)$. One class of tangent vectors leaving the second component of $\boldsymbol{E}$ invariant is generated by the following:

$$\left(0\ ,\ h_1(\rho_1, x_2, y_2)\frac{\partial}{\partial x_2}\right) \quad \text{and} \quad \left(0\ ,\ h_2(\rho_1, x_2, y_2)\frac{\partial}{\partial y_2}\right),$$

namely those generating right-transformations that leave ρ_1 invariant altogether. (Here the h_i are arbitrary functions of ρ_1, x_2, y_2.) Another class is formed by vectors generating left-transformations that leave the second component unchanged:

$$\left(f_1(\zeta_1, \zeta_2)\frac{\partial}{\partial \zeta_1}\ ,\ 0\right).$$

Here f_1 is an arbitrary function of two variables. The third class of vectors generate left–right transformations, interacting in such a way that their *combined* action leaves $\boldsymbol{E}$'s second component invariant:

$$\left(-\zeta_1 f_2(\zeta_1,\zeta_2)\frac{\partial}{\partial\zeta_2}\ ,\ f_2(H,\rho_1)\boldsymbol{\alpha}\right) \quad \text{and} \quad \left(-\zeta_2 f_3(\zeta_1,\zeta_2)\frac{\partial}{\partial\zeta_2}\ ,\ f_3(H,\rho_1)\rho_1\frac{\partial}{\partial\rho_1}\right).$$

Again the f_i are arbitrary functions, and $\boldsymbol{\alpha}$ a vector field on $\mathbb{R}^3$ such that $\boldsymbol{\alpha}\rho_1 = H$. Applying these tangent vectors to the map $\boldsymbol{E}$ and projecting to the first component, the reduced tangent space (to the map $\boldsymbol{E}$) becomes

$$\left\{h_1\frac{\partial H}{\partial x_2} + h_2\frac{\partial H}{\partial y_2} + f_1(H,\rho_1) + f_2(H,\rho_1)\boldsymbol{\alpha}H + f_3(H,\rho_1)\frac{\partial H}{\partial\rho_1}\right\} \subseteq \mathcal{E}_{\rho_1,x_2,y_2},$$

where $\mathcal{E}_{\rho_1,x_2,y_2}$ is the set of functions depending on ρ_1, x_2, y_2. Observe that this is not, indeed, an ideal over $\mathcal{E}_{\rho_1,x_2,y_2}$.

To complete the discussion of this example, we note that the first two terms together span the tangent space related to the right-transformations only. Since in the planar reduction example this alone was enough to get to a codimension 1 singularity, the present tangent space has at most codimension 1. In fact, it is easy to see that x_2 is not in the tangent space, so that the codimension is indeed 1. Normal form and versal unfolding are identical to the previous example; see Fig. 1.1 for bifurcation curves.

This completes the *qualitative* analysis. We now have a versal model of the system; it remains to compute coordinate transformations inducing the actual (Birkhoff-normalized) system from the versal model. One ingredient, Kas and Schlessinger's algorithm, may be moulded to fit the present situation quite straightforwardly, the only technical complication being the step from the reduced tangent space back to the actual tangent space of the mapping. As in the planar reduction case, Kas and Schlessinger's algorithm requires solutions to the infinitesimal stability equation. To compute these efficiently is more difficult.

In contrast to the planar reduction method, which uses right-transformations as equivalences and consequently yields an ideal as tangent space, currently the tangent space has a more complicated structure, since it is a combination of an ideal and a module over an algebra. Our initial idea was that once a suitable generalization of a Gröbner basis for algebras (instead of ideals) was found, integration of the various parts would be straightforward. The required generalization was known in the literature as *canonical subalgebra basis* [Stu93, Stu96] or *SAGBI basis* [Vas98].

1.1.3 Standard bases

However, applying these ideas to our example of the left–right tangent space turned out to be difficult. For some time we tried to resolve the difficulties in an *ad hoc* manner, resulting in several almost-identical algorithms and constructions.

From these, we could finally pinpoint the similarities and differences between the various constructions ranging from Gröbner bases, standard ideal bases, canonical subalgebra bases to analogous bases for left–right tangent spaces. All of these constructions could be put in a general framework, built around two central objects called *standard maps* and *standard bases*. Viewing these constructions as instances of a more abstract construction greatly clarified the situation. It suggested transparent notation, and eliminated the near-similar proofs.

A central role is played by the *standard map theorem*, theorem 6.10. It was inspired by Greuel's proof [GP88] of the ordinary Gröbner basis theorem, which exploits an idea by Schreyer [Sch91] for computing the module of syzygies of ideal generators. In contrast to the usual proofs, Greuel's proof uses the ideal structure only superficially, which facilitated the generalization to other algebraic structures. Another object borrowed from Greuel that, with a small modification, proved very useful, is the *normal form map*. This map plays the role of a 'lazy' division algorithm, analogous to the normal form algorithm for Gröbner bases. (Incidentally, this map is unrelated to Birkhoff's normal form procedure.) The map is required to satisfy only mild conditions, and exists under very general assumptions on the base ring and term ordering: For the polynomial ring, it exists whenever the term order is a well-order, whereas for the formal power series ring, the term order should be a reversed well-order. For details also see [Lun99b]

1.2 Sketch of the results

We briefly present our results, splitting the presentation into two parts. First we discuss the results obtained in applying the algorithmic methods to the spring–pendulum in chapters 2 and 3. Second we discuss the algorithmic methods themselves, as far as they are new.

1.2.1 Reduction methods

Both reduction methods for Hamiltonian systems around an equilibrium point and close to resonance, elaborated in chapters 2 and 3 respectively, have the same goal, namely describing the local dynamics of the system.

Remark 1.1. To be precise, we describe the dynamics of the *Birkhoff normalized* $\mathbb{S}^1$-symmetric system, which differs from the original system by a smooth coordinate transformation and a flat (generally) non-symmetric perturbation. The perturbation problem involved here is ignored in this study, for some remarks and references see the preface. Since we focus on an integrable approximation, we do not find chaotic dynamics in our final model system. However, since the the real system is a flat perturbation of the model system, the difference is extremely small for modest energies (i.e., for modest deviations from the equilibrium) so that our approximation is very good. This conclusion is confirmed by numerical

experiments. In two degree of freedom systems we can say even more, see the introduction to chapter 2.

Since both reduction methods differ considerably in their approach, but still have the same aim, it is interesting to compare their results. We chose to apply both methods to the spring–pendulum system in $1:2$ resonance. This particular resonance was chosen because both methods are able to handle it. In chapters 2 and 3 we do a part of the analysis for other resonances as well, but only the $1:2$ case is carried out fully.

1.2.2 The planar reduction method

Our spring–pendulum model (see Fig. 2.1 in section 2.3.1) has various physical parameters, like pendulum length and mass, spring constant *etc.* For simplicity we use the coefficients a_i of the Taylor expansion of the system's Hamiltonian as parameters, see (2.7). The Hamiltonian has two degrees of freedom, and can be brought in the form

$$H^0(x_1, y_1, x_2, y_2) := \frac{x_1^2 + y_1^2}{2} + a_1 \frac{x_2^2 + y_2^2}{2} + h.o.t. \tag{1.4}$$

Here the x_i are configuration variables and the y_i momentum variables. The parameter a_1 determines the frequency ratio of the two harmonic oscillators of the quadratic truncation (or linear truncation of the vector field). We shall assume we are close to the $1:2$ resonance, i.e., $a_1 \approx 1/2$.

Next, we apply the Birkhoff normalization procedure, and we truncate. The result is an $\mathbb{S}^1$ symmetry, and a conserved quantity λ. Regarding λ as a parameter and dividing out the symmetry, we get a planar system in the variables x, y. The relation between the original variables x_1, y_1, x_2, y_2 and the planar variables x, y is as follows: On the section $y_2 = 0$ we have $x = x_1$ and $y = y_1$, modulo higher order terms. A singularity theory normalization of the planar system, as explained in Sect. 1.1, yields the following versal family:

$$H^u(x, y) = x(x^2 + y^2) + u_1 x + u_2 y^2, \tag{1.5}$$

the $\mathbb{Z}_2$-symmetric hyperbolic umbilic D_4^+, see Proposition 2.13. Here u_1 and u_2 are deformation parameters. (The $\mathbb{Z}_2$ symmetry, originating from the system's reversibility, and acting as $(x, y) \mapsto (x, -y)$, is the reason that no deformation term $u_3 y$ is needed.) The family defines a Hamiltonian system in x and y which is equivalent (i.e., conjugate modulo time reparametrizations) to the planar reduction.

We here reproduce the bifurcation diagram, Fig. 2.5, in Fig. 1.2. It depicts the phase diagrams that occur in the planar system (1.5); the shaded parts correspond to inaccessible regions either in the phase or parameter plane. (They occur because of singularities in the coordinate transformations; points in the inaccessible regions correspond to imaginary values of parameters or phase variables.)

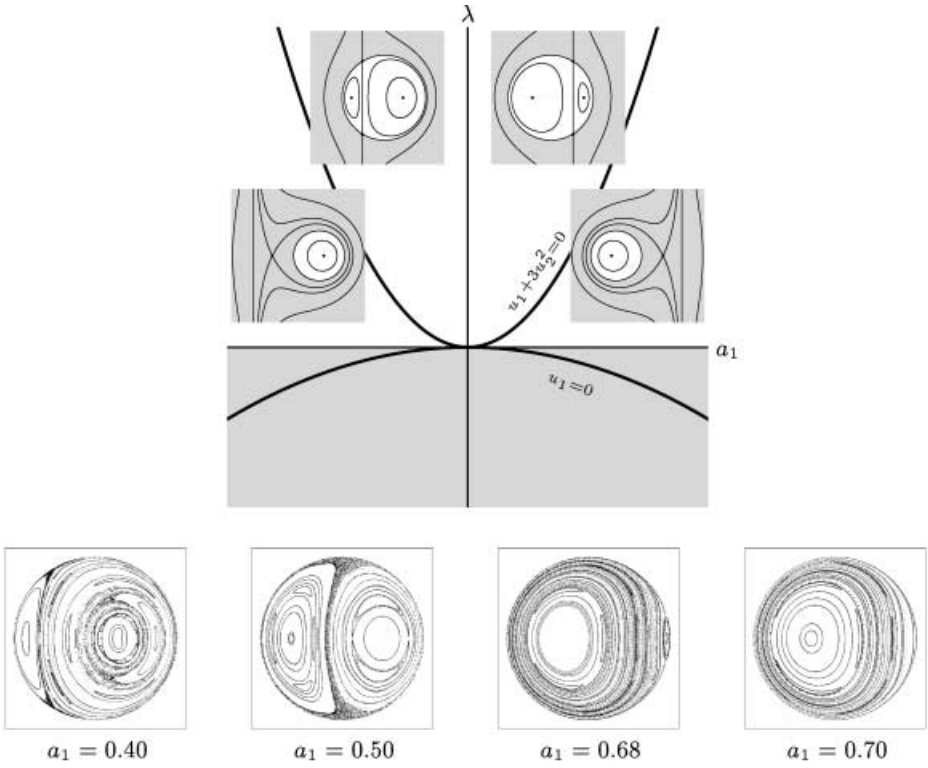

Fig. 1.2 Bifurcation diagram for the model (1.5), and numerical iso-energetic Poincaré maps. Grey areas correspond to non-physical states or parameter settings. Note the similarities between the bifurcation diagram and the Poincaré maps, as well as the effect of nonintegrability in the latter.

In the family (1.5) two curves of bifurcations occur, $u_1 = 0$ and $u_1 + 3u_2^2 = 0$ (see Sect. 2.3.3). The former lies in an inaccessible region, whereas the latter corresponds to a line of Hamiltonian pitchfork bifurcations. With the help of explicit expressions for the normalizing transformations, we can pull back this curve to original parameters. The result is the following bifurcation curve:

$$\lambda = \frac{(1-2a_1)^2}{64a_2^2} + \frac{(a_2^2-a_4)(1-2a_1)^3}{128a_2^4} + O((1-2a_1)^4), \tag{1.6}$$

see Proposition 2.18. Here the a_i are coefficients of the original Hamiltonian (1.4) or (2.7), and λ is the conserved quantity. An expression for the latter in original phase space variables can be computed using the Birkhoff normalization procedure; see (2.8). Combining (1.6) and (2.8), the bifurcation equation can be checked by numerical simulations of the Poincaré map. The results may be found in table 2.2 and Fig. 2.3.

The final step involves a versal model under a restricted equivalence class, the so-called BCKV-restricted morphisms; see [BCKV93, BCKV95, BHLV98] as well as Sect. 2.2.7. This class distinguished three levels of variables, namely phase space variables, distinguished parameters and ordinary parameters. The distinguished parameters are parameters for the final model, but originate from phase space, as conserved quantities. These morphisms allow transformations of distinguished parameters to depend on both kinds of parameters, but disallow those of ordinary parameters to depend on distinguished ones. This ensures that reparametrizations applied on the reduced family may be pulled back to the original phase space without mixing parameters and phase space variables. This results in a model given in Proposition 2.16. It gives rise to the same qualitative behavior, i.e., bifurcations, but with an additional nondegeneracy condition; see also remark 2.17 in Chapter 2.

Other resonances The planar reduction method cannot be used for resonances $p : q$ with q odd. The reason is that the iso-energetic 3-torus on which the system lives has two singular points, corresponding to simple periodic motion. When reducing this to the plane, one of these is mapped to a circle, here referred to as the *singular circle* (see remark 2.5 and Sect. 2.3.3). It is then essential that the function is smooth at this circle, which only happens for q even (see Proposition 2.4).

Having done the 1:2 case, the next case to consider is the 1:4 resonance. Again from Proposition 2.4 one can (correctly) guess that the versal family has a central singularity of the form

$$\alpha(x^2 + y^2)^2 + \beta x(x^2 + y^2)^2. \tag{1.7}$$

With respect to equivalence via $\mathbb{Z}_2$-equivariant planar right-transformations, this function has infinite codimension. This seems to imply that the system in $1 : 4$ resonance is highly degenerate, and that small perturbations can spawn many qualitatively different kinds of dynamics. This is not the case: Both the odd-resonance problem and this high-codimension phenomenon result from perturbing within the class of $\mathbb{Z}_2$-invariant planar functions.

If a perturbation is applied to the *original* system, then after Birkhoff normalization this perturbation is 'projected' inside the class of functions generated by the $\mathbb{S}^1$-invariants $x^2 + y^2$, λ, and one of

$$(x^2 - y^2)(x^2 + y^2)^A(x^2 + y^2 - \lambda/C)^B \quad \text{and} \quad x(x^2 + y^2)^A(x^2 + y^2 - \lambda/C)^B.$$

(Here A, B and C are integers depending on p and q, see Proposition 2.4.) To require (1.7) to be versal with respect to arbitrary ($\mathbb{Z}_2$-symmetric) planar perturbations is to ask for more than we need. The class of equivalences, arbitrary $\mathbb{Z}_2$-equivariant right transformations, is also larger; however, the net effect is still that (1.7) has infinite codimension with respect to this class of perturbations.

In the next section we consider another reduction method, which uses equivalences that respect the $\mathbb{S}^1$-symmetry. It turns out that this method does find finite codimension models for higher order resonances.

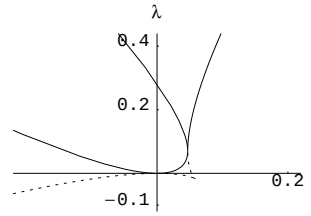

Fig. 1.3 Bifurcation curves for (1.8), with parameters $a = \frac{1}{4}$ and $b = \frac{1}{3}$.

1.2.3 The energy–momentum map method

As with the planar reduction method, we use the coefficients of the Taylor expansion of the spring–pendulum system (equation 2.8) as basic parameters, and again assume that we are close to the 1 : 2 resonance, i.e., $a_1 \approx 1/2$. As normal form for the energy–momentum map we find

$$(H^\mu, \lambda) = \left(\frac{1}{2}\rho_1 + \left(\frac{1}{4} + \mu\right)\rho_2 + a\psi + b\rho_1\rho_2 \quad , \quad \frac{1}{2}\rho_1 + \frac{1}{4}\rho_2\right), \tag{1.8}$$

see (3.12). Here λ is the conserved quantity associated with the $\mathbb{S}^1$ symmetry, which in this case is just the quadratic part of H. In (1.8), a and b are coefficients, μ the deformation parameter, also called the *detuning parameter*, and ρ_1, ρ_2, ψ are three $\mathbb{S}^1$-invariants that form a Hilbert basis for the $\mathbb{S}^1$-symmetric functions. They have a relation which is implicit from the following connection with the coordinates x, y used in the planar reduction case:

$$\rho_1 = x^2 + y^2, \quad \rho_2 = 4\lambda - 2(x^2 + y^2), \quad \psi = 2x(2\lambda - x^2 - y^2).$$

The bifurcations of this model, i.e., the μ values for which the number of relative equilibria changes, form a parabola and a cubic curve in the λ, μ plane; see Fig. 3.4 which is reproduced here as Fig. 1.3. Figure 1.4 displays the bifurcation sequence for fixed, small and positive λ, and increasing μ. (We used the Poincaré section transversal to the long periodic orbit; for the other Poincaré section see Fig. 3.5.) The parameter μ is related to the quantity $1 - 2a_1$, which measures the deviation from the 1 : 2 resonance. Three bifurcations occur as μ increases, hence its name of 'detuning parameter'. Two of those are pitchfork bifurcations corresponding to the parabola and occur for small μ. The other is a saddle–node bifurcation and occurs for finite μ, in the limit as λ tends to 0. To pull back the bifurcation curve to original coordinates, the λ-level at the bifurcation points need to be computed, as in contrast to the planar reduction method, λ-levels are not preserved by the left–right transformations. The result is

$$\lambda = \frac{(1-2a_1)^2}{64a_2^2} + \frac{(2a_2^2 + 6a_3 - a_4 - 2a_5)(1-2a_1)^3}{128a_2^4} + O(|1-2a_1|^4) \tag{1.9}$$

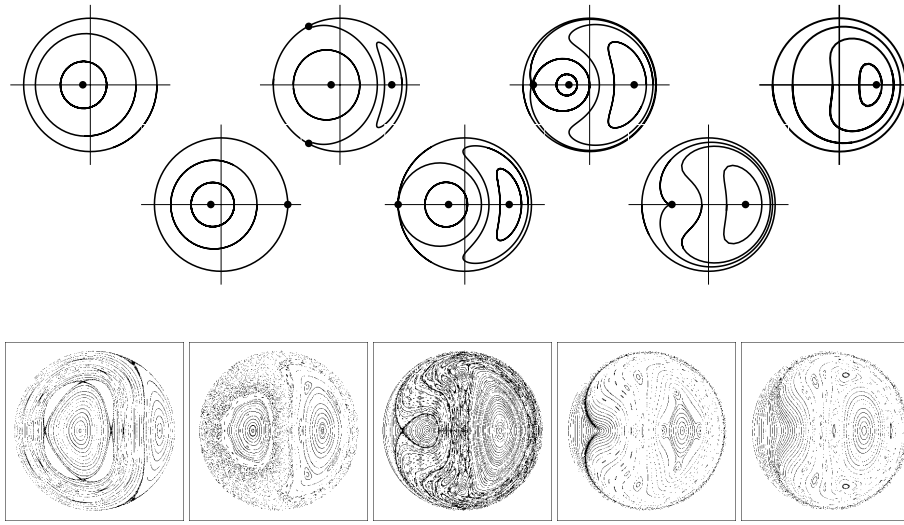

Fig. 1.4 Bifurcations of the energy–momentum map normal form around the 1 : 2 resonance for small increasing μ (left to right), crossing the solid curves of Fig. 1.3 three times, resulting in four structurally stable phase diagrams (top row). Below are some corresponding Poincaré sections obtained by numerical integration.

for the pitchfork bifurcation; see Sect. 3.4.3. Here λ is the integral of the system, see (2.8). The saddle–node bifurcation occurs outside the origin, and for $O(1)$ values of the detuning parameter $1 - 2a_1$; see (3.19) in Sect. 3.4.1, which means that a formal power series approximation of a coordinate transformation cannot be used to approximate the bifurcation curve in original parameters.

Remark 1.2. The saddle–node bifurcation away from $(\lambda, \mu) = (0, 0)$, however relevant it may be for the physical model and numerical simulations under study, is outside the present theoretical consideration. A more careful analysis of the model will be needed to study this bifurcation.

As can be seen from (1.8), the versal model has 1 parameter, so it has codimension 1. It therefore seems that this model has lower codimension than the planar model of the previous section. However, this difference is superficial and due to accountancy only. Indeed, the planar reduction method regards λ as a (distinguished) parameter, whereas with the present energy–momentum method λ is part of the map. (Earlier computations on the energy–momentum map used only right-transformations, which resulted in a codimension 2 model, see [Dui81].)

Other resonances The methods used for the 1:2 resonance can be applied more generally. The 1:4 resonance (see Sect. 3.3.3) results in a versal family with 2 parameters (or 3 if λ is counted as a parameter). Apart from small differences

in the formulas for the nondegeneracy conditions, the results agree with those in [Dui84].

The computations for the 1:3 resonance are rather involved. The complexity is evidenced by the lengthy expressions for the nondegeneracy conditions; see Sect. 3.3.2. For sketches of the bifurcation diagrams for both of these resonances, as well as for the cases $p : q$ with $|p| + |q| > 4$, e.g., see [Dui84, GMSD95].

Other interesting resonances are the 1:1 and 1:-1 cases. In contrast to higher order resonances, the sign (referring to the positive-definite and indefinite cases, respectively) makes a great difference here. In [Mee85] the 1:-1 resonance is analyzed in detail. Cotter [Cot86] analyzes the semisimple 1:1 resonance. It would be interesting to apply our methods to these cases, in order to obtain explicit bifurcation curves. This is relevant for systems like the restricted three-body problem around Routh's mass ratio, see, e.g., [DH68, Mee85, MH92, MS71, Rou75].

Remark 1.3. An interesting example to apply the present techniques to, is the restricted three-body problem. Like the spring–pendulum, also this system has two degrees of freedom. The main difference with the former is that it has a hyperbolic instead of elliptic equilibrium, leading to different 'allowed zones' in the bifurcation pictures.

1.2.4 Standard bases

Gröbner bases enabled the algorithmic treatment of polynomial ideals. We now briefly sketch the ideas behind it, in order to give a heuristic introduction to standard bases and to indicate the relationship between Gröbner and standard bases.

Given any order on the monomials (satisfying some conditions that we ignore for now), a Gröbner basis is a basis $\{h_1, \ldots, h_n\}$ for an ideal $I = \langle h_1, \ldots, h_n \rangle$ such that every monomial that occurs as *leading monomial* of an element of I, according to the monomial order, is a multiple of the leading monomial of some h_i. In a precise sense, a Gröbner basis is to its ideal what a matrix in row-echelon form is to its image, when regarded as a linear operator. To be specific, for a matrix A in row-echelon form it is easy to determine the codimension of its range, and for a Gröbner basis it is similarly easy to determine the codimension of the corresponding ideal regarded as a real vector subspace of the ring in which it lives. Also, for such a matrix, solving the equation $Ax = b$ for x is straightforward, and this corresponds to the existence of a so-called *normal form map* (or algorithm) solving a similar equation for ideals; in particular, it solves the ideal membership problem. Standard bases are a generalization of Gröbner bases to algebraic structures other than ideals – and to a lesser extent, also to other base rings than the polynomial ring. They are best explained using Fig. 6.3 which is reproduced here as Fig. 1.5. For the case of Gröbner bases, R is the polynomial ring, $M = R^n$ and Ψ is the R-module homomorphism given by $(a_1, \ldots, a_n) \mapsto \sum a_i h_i$, so that the image of Ψ is the ideal I. The map $\widetilde{\Psi}$ maps $(a_1, \ldots, a_n)$ to $\sum a_i \,\mathsf{LT}\, h_i$; here $\mathsf{LT}\, h_i$ is the leading term of h_i, so that the image

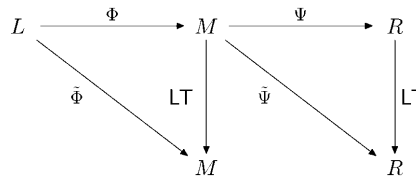

Fig. 1.5 Diagram for the standard map theorem. Under computable conditions on Φ and Ψ, the sequence $L \xrightarrow{\Phi} M \xrightarrow{\Psi} R$ is exact and Φ, Ψ are standard maps.

of $\widetilde{\Psi}$ are all multiples of leading monomials of the ideal generators h_i. Now the question of $\{h_i\}$ being a Gröbner basis can be rephrased as: Is it true that

$$\mathsf{LT}\,\mathrm{Im}\,\Psi = \mathrm{Im}\,\widetilde{\Psi} \quad ?$$

(Here LT applied to a set is the span of the $\mathsf{LT}\,f$, where f ranges over the set.) If equality holds, the map Ψ is called a *standard map*, and the basis $\{h_i\}$ a *standard basis*. For ideal bases, the question whether the associated map Ψ is a standard map is settled by looking at S-polynomials (for *syzygy polynomials*), combinations of ideal generators that cancel leading terms. In other words, S-polynomials are the images under Ψ of elements of $\ker\widetilde{\Psi}$. The map Φ is constructed so that its image is the kernel of Ψ; in other words L is the (first) syzygy module of $(h_1, \ldots, h_n)$. It was Schreyer's insight that including the syzygy module L considerably simplifies the proof of the main result in Gröbner basis theory. By making the maps Ψ and Φ explicit it is easy to abstract from the ideal structure, and apply the same idea to other algebraic structures. In essence, this is the idea of the standard map approach.

1.3 Discussion

We present a discussion on our results and methods.

1.3.1 Discussion – reduction methods

In the previous sections the application of both the planar reduction (PR) method and the energy–momentum (EM) method to the spring–pendulum in 1 : 2 resonance has been summarized. At this point therefore, we may try to compare both methods.

Both methods arrive at precisely the same asymptotic expansion of the bifurcation curve for the Hamiltonian pitchfork bifurcation, (1.6) and (1.9), which also agrees well with the numerical results of table 2.2, Sect. 2.3.3.

The polynomial normal forms the methods arrive at are rather different. The PR method arrives at a homogeneous third order polynomial as planar Hamiltonian, which was easy to analyze. In contrast, the energy–momentum

method yielded a map with a quadratic and a quartic polynomial as components. This model has a more complicated bifurcation structure than the planar model. However, as stated before, the additional bifurcation (a saddle–node bifurcation) occurs away from the origin in parameter space.

Besides yielding a simpler model, the calculations associated to the PR method are simpler than those of the EM method, because the former employs (equivariant) right-transformations as equivalences in the second reduction stage, resulting in a tangent space with an ideal structure. The EM method also uses left-transformations, yielding a much more complicated tangent space. Technical complications also arise from the additional reduction step, used for reducing from a tangent space of maps to $\mathbb{R}^2$, to one of functions to $\mathbb{R}$. The de-reduction step necessary when computing the reparametrization explicitly is not needed for the PR method. Finally, the more general transformations allowed in the EM method deform the H_2-level sets and require calculation of the inverse transformation, in order for the bifurcation curve to be found (see Sect. 3.4.3).

The EM method is able to handle any resonance. In Chap. 3 the $1:2$ resonance is analyzed in detail, and some calculations for the $1:3$ and $1:4$ case have also been done. The PR method is less suited for analyzing resonances with odd denominator, because square roots turn up in an essential way. In addition, the symmetry resulting from the Birkhoff normalization is not fully exploited, leading to a higher codimension. Indeed, for the $1:4$ resonance it is already infinitely high. Interestingly, for the $1:2$ case the codimensions are equal for both reduction methods.

The PR method, and especially the final BCKV normal form, seems better suited to de-reduction to the full phase space than the EM method. The reason is that the latter mixes the integral of motion H_2 with the Hamiltonian H in the left-transformations on the energy–momentum map, whereas BCKV-restricted morphisms do not. For the organization of Hamiltonian level sets on levels of H_2 this makes no difference. It does necessitate an additional computation to pull back the bifurcation curves in terms of H_2. Perhaps more importantly, if one is interested in the organization of the Hamiltonian's level sets in the full phase space, the extra transformations need to be taken into account when drawing conclusions for the real system from the model.

1.3.2 Discussion – standard bases

Standard bases arose in the course of our research, as a natural way to bring several very similar results on common denominator. The result was a unifying proof of the basic results on Gröbner bases, based on a proof due to Schreyer and Greuel, which lent itself immediately to the necessary generalizations.

We almost always work in the ring of truncated formal power series. Results for the formal power series ring are also given (see Sect. 6.5), and involve only an existence proof of the normal form map in this context. We give some remarks on the case of the rational function ring, where Mora's algorithm takes the place of the normal form map. The structures we apply the standard map approach to,

are ideals, modules and algebras, for which the results are already well-known. For example, what we call standard bases for algebras are known as *canonical subalgebra bases* or *SAGBI bases* in the literature. Our main application, standard bases for left–right tangent spaces, is surely new. It should be easy to adapt the approach to other situations as well.

For example, the tangent space used by Golubitsky and Schaeffer [GS85] of unfoldings $g(x, \lambda)$, where $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, consists of all germs of the form

$$a(x, \lambda)g + b(x, \lambda)\frac{\partial g}{\partial x} + c(\lambda)\frac{\partial g}{\partial \lambda}. \tag{1.10}$$

Here a, b, c are arbitrary functions, and note that c does not depend on x. Because of this, this tangent space is not an ideal in the ring of functions in x, λ. In [GS85], the codimension of the vector space (1.10) is computed by noting that one may restrict to the space of truncated formal power series, for a suitable truncation degree, reducing the question to linear algebra. The standard map approach is a systematic and efficient method for solving this linear algebra problem.

1.4 Outline

This work is split into two parts. In Part I we discuss the two reduction methods for Hamiltonian systems considered here, and apply the algorithmic methods to compute exact bifurcation curves in original coordinates,comparing these with numerically obtained phase pictures. The dynamical system we consider is a two-degree-of-freedom spring–pendulum around $1 : 2$ resonance. We decided to focus on one specific system, around a single resonance that can be analyzed by both methods, in order to make comparisons. The methods are general, however, and throughout generality is retained in the formulations as much as possible.

After these applications, the theory behind these computations is developed in Part II. Much well-known material is summarized in some detail, such as Hamiltonian mechanics, Birkhoff normal forms and singularity theory. We retain an algorithmic focus throughout, but especially in Chap. 4 about Birkhoff normal forms, and Chaps. 6 and 7 about Kas and Schlessinger's algorithm and the infinitesimal stability equation. We now give a short overview for each chapter.

Chapter 2 deals with the planar reduction method, involving first Birkhoff normalization, then a symmetry reduction to the plane using the $\mathbb{S}^1$ normal form symmetry, and subsequently planar $\mathbb{Z}_2$-equivariant singularity theory. A versal model of the planar unfolding is computed, with its bifurcation curves, which are pulled back to the original parameter space and compared to numerical results. Basically, this follows our paper [BHLV98].

Chapter 3 deals with the energy–momentum map method. This too starts with a Birkhoff normalization, but then it immediately applies singularity theory to the energy–momentum map, with equivalences defined by left–right transformations. Again a versal model is computed, its bifurcation diagram pulled back, and results compared to numerical pictures.

Chapter 4 summarizes some Hamiltonian mechanics and the Birkhoff normal form. We quote Birkhoff's original result, as well as more modern version which allows normalization around resonances. In line with the algorithmic emphasis we make a comparison between several algorithms for calculating the Birkhoff normal form, and propose one of our own.

Chapter 5 introduces the necessary singularity theory. As explained before, we use both right-transformations as well as left–right transformations. To explain the ideas we first give the finite-dimensional version, with a smooth Lie group acting on a smooth manifold. Then we quote analogous results for the case of smooth reparametrizations acting on function spaces, where technical complications show up that necessitate the use of the Mather–Malgrange preparation theorem. In this chapter we prove some smooth results, but this is not given great emphasis as in the applications we always use truncated formal power series. However, the results are important in order to show these finite expansions are indeed the 'shadow' or 'projection' or something 'really out there' in the form of smooth transformations.

Chapter 6 develops the standard map approach to Gröbner bases. Motivation for this approach has been given in this introduction, and is provided by Chap. 7 which applies the results obtained here. The chapter starts with a quick overview of Gröbner basis, in a slightly different perspective facilitating the step to more general situations. We proceed to give the abstract underpinnings of the standard map approach, listing the properties and constructions involved, and proving the standard map theorem 6.10. We proceed to prove the basic facts of Gröbner bases using this approach, and doing the same for canonical subalgebra bases and a few others, culminating in the standard basis theorem for left–right tangent spaces as used in the energy–momentum reduction method.

Chapter 7 shows how to combine Kas and Schlessinger's algorithm and the results from the previous chapter to perform the calculations of Chaps. 2 and 3. For the planar reduction method this is straightforward, and has been published in [BHLV98]. The complications for the energy–momentum reduction method are twofold. Most of the complications are related to the algebraic structure of the left–right tangent space, and are dealt with in Chap. 6. The other complication is technical in nature: The calculation of the tangent space involves a reduction from maps to $\mathbb{R}^2$ to functions, and to complete the calculation this reduction has to be undone. This involves reading the story of Chap. 3 backwards, and results in a long calculation.

2 Method I: Planar reduction

We apply the planar reduction method to a general two degree of freedom system with optional symmetry, near equilibrium and close to resonance. As a leading example the spring-pendulum close to 1:2 resonance is used. The resulting planar model is computed explicitly, and the bifurcation curves obtained are compared to numerical simulations.

2.1 Introduction

Our goal is to obtain a description of the dynamics and bifurcations of a Hamiltonian system near equilibrium. To reach this goal, we use in this chapter the so-called planar reduction method, which we describe below. The result is a polynomial Hamiltonian model system living on the plane. This system is easy to analyze, yielding a qualitative description of the original system. In particular, bifurcation curves are easy to find. What is new in the current approach is that in each step towards the final polynomial model, the simplifying transformations are computed explicitly. This allows us to pull back the final bifurcation curves to the original parameter- and phase-space, so that quantitative results for the bifurcations of the original system are obtained. These results can subsequently be checked against numerical simulations of the iso-energetic Poincaré map, of which the planar model is an integrable approximation (see e.g. [BCKV93]). The agreement of numerical data and the pulled-back bifurcation curve is good, especially for small excitations.

The reduction methods, used here and in the next chapter, consist of three parts: Birkhoff normalization, symmetry reduction, and singularity theory. In each of these stages the coordinate transformations are explicitly computed, and especially in the first and final stage this is rather involved. For these calculations we summon the computer's help, using algorithms described and developed in later chapters. Algorithms for Birkhoff normalization are described in Chap. 4, whereas Chaps. 6 and 7 deal with the computation of the transformations for the singularity theory stage.

The reduction methods can be applied to systems with an equilibrium at the origin, that have a nondegenerate quadratic part exhibiting a single resonance between two of the n degrees of freedom. Optionally, the system may have (e.g. discrete) symmetries. The system is first subjected to the Birkhoff normalization

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procedure. The result is a formal coordinate transformation, and a normalized system with a $\mathbb{T}^{n-1}$ torus symmetry. By Noether's theorem (which holds for Lagrangian systems; see e.g. [AM78] for the Hamiltonian version), the 1-parameter continuous symmetries are related to conserved quantities, in this context also called *momenta*, see [CS85]. After applying the Birkhoff procedure, $n-1$ of such independent conserved quantities can be found explicitly.

The large symmetry group and related conserved quantities imply that the system is integrable, but generically an n degree of freedom system ($n \geq 2$) is not [BT89]. In fact, the formal transformation lifts to smooth coordinate transformations by a theorem of Borel and Schwarz [Bro81, Dui84, GSS88], but these do not form conjugations. However, they do transform the system to a *perturbed* version of the normalized system. This perturbation is flat in the phase variables, so that formally conserved quantities are actually adiabatic invariants, and solution curves of the integrable system stay close to those of the original system for a long time.

For two degrees of freedom the situation is even better. By KAM theory, there exist a fat Cantor set of tori with parallel dynamics. On this part of phase space, a smooth conjugation with the integrable system does exist. The KAM tori prevent chaotic solution curves from wandering through phase space, so that even these solutions stay within a bounded distance from the integrable system's tori for ever. This provides a justification for using the integrable approximation to study the full system; see also [LL92]. From here on, therefore, we shall ignore the flat perturbation.

Our aim in this chapter is to obtain formulas for bifurcation curves up to a certain degree. Therefore, we also need the Taylor series of the normalized system up to certain degree only. By performing the iteration in Birkhoff's procedure a finite number of times, we obtain a smooth (in fact, polynomial) transformation and *approximate* symmetries, which we make exact by truncating the system.

The second part, symmetric reduction, is well-known and goes back to the Kepler problem, see e.g. [AM78, Bro79, CS85, CB97, Mee85, Tak74b]. The idea is to divide out the symmetry, and regard the associated conserved quantities as parameters (also called *integrals*), a procedure known as *orbit space reduction*. Sometimes this reduction is done on the entire phase space, and sometimes on each leaf of the foliation defined by the levels of the integrals. Care has to be taken when the topological type of the leaf depends on the value of these integrals, and when the symmetry group does not act freely (i.e., when some points have nontrivial isotropy group); see [AM78, CS85] for details. The result is a reduced Hamiltonian system with one degree of freedom, and whose dynamics coincides with the projection of the dynamics of the original system onto the orbit space.

At this point the planar reduction method, and the energy-momentum method of Chap. 3 start to diverge: Both the reduced system, as well as its subsequent treatment with singularity theory, are different. The reduction method of this chapter now applies a symplectic transformation that reduces the system to the plane, which gives the method its name.

The last stage consists of normalizing the planar system by arbitrary right-transformations. Such transformations are not symplectic, and therefore do not yield conjugations. However for 1 degree of freedom systems they do yield *equivalences*, i.e., conjugations modulo smooth time-reparametrizations. In particular, bifurcations are preserved, as they involve equilibria in the reduced system. The result is a polynomial model with parameters, together with transformations and reparametrizations that connect it with the original system.

In the first part of this chapter we describe the method in general terms, after which it is applied to the leading example of the spring-pendulum, a two degree-of-freedom system with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ spatial and reversing symmetry.

Large parts of this chapter have been published in [BHLV98, BLV98]. The planar reduction method as used here was introduced in [BCKV95, BCKV93]. It stands in a long tradition, see e.g. [Mee85] for a historical overview. We here also mention the method described in [Dui84] which will receive full treatment in Chap. 3; see also [VvdM95].

2.1.1 BCKV-restricted morphisms

An important aspect of the equivalence transformations used to simplify the planar system, is their treatment of the *distinguished parameter* λ, the conserved quantity resulting from Birkhoff normalization. It is a parameter to the planar system, but a special one since it depends on the original phase variables, hence the adjective *distinguished*. This special nature of λ is reflected in the class of reparametrizations allowed on the planar system: It is required that the zero-level of λ is preserved, and that reparametrizations of ordinary parameters do not depend on λ; see remark 2.6 and [BCKV95, BCKV93] for more details. Theorem 5.16 implements these restrictions, yielding for the case of the $1:2$ resonance a normal form

$$x(x^2+y^2) + (\lambda_1 + u_1)x + (\lambda_2 + u_2)y^2.$$

Here λ_i and u_i are distinguished and ordinary parameters, respectively. One consequence of the theorem is that a versal deformation requires at least two distinguished parameters. For our application we have only one at our disposal, a problem which is resolved by the *path formulation*, see [BCKV93, FSS98, Mon94]. The resulting normal form is given in Proposition 2.16, where one of the distinguished parameters depends explicitly on the other parameters, so that it traces out a path through the parameter space of the versal deformation.

2.2 Details of the planar reduction method

In this section we give a detailed outline of the planar reduction method. The method is applied to the spring-pendulum around the $1:2$ resonance in Sect. 2.3.

Table 2.1 Overview of the planar reduction of the spring-pendulum in 1 : 2 resonance

Section:		2.2.3	2.2.4	2.2.5	2.2.6	2.2.7
Context:	Original	Birkhoff normal form	Planar reduction	Central normal form	Versal deformation	BCKV normal form
System:	H^0	H^n	H^r	H^c	H^u	H^B
Phase space:	$\mathbb{R}^4$	$\mathbb{R}^4$	$\mathbb{D}^2$	$\mathbb{R}^2$	$\mathbb{R}^2$	$\mathbb{R}^2$
Coefficients:	$a_i, i \geq 1$	$b_i, i \geq 2$	$b_i, i \geq 2$	d_i	—	a_i
Parameters:	—	b_1	b_1, λ	c_i, λ	u_1, u_2	λ, b_1
Symmetry:	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{S}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

During the reduction, the system, the parameters it depends on, the phase space on which it lives, and its symmetry change several times. A summary is given in table 2.1. Using this as a guide, we start this section by outlining the reduction procedure leading to the BCKV normal form. In 2.2.2 we give some notation that will be used in the sequel, after which we begin discussing the reduction method proper.

2.2.1 Overview

The starting point is a Hamiltonian H^0 with an equilibrium at the origin. It is supposed to be close to some resonance of the form $p : q$, and to depend on several coefficients a_i. Optionally, the system may be invariant (or reversing) under some symmetry group Γ, which is supposed to respect the symplectic structure. (Table 2.1 shows the symmetry group relevant for the spring-pendulum example in the various stages during reduction.)

The first step is to apply the Birkhoff procedure around the resonance, resulting in a system H^n which has acquired a (formal) $\mathbb{S}^1$-symmetry. This step singles out a *detuning parameter* denoted by b_1, which measures the deviation from the resonance around which the Birkhoff procedure is performed. For convenience, the other coefficients are now denoted by $b_2, b_3, \ldots$, and depend on the a_i.

The system H^n has two independent conserved quantities: H^n itself, and the formal integral λ conjugate to the cyclic variable associated to the $\mathbb{S}^1$ symmetry. In the cases we consider, this integral is equal to the quadratic part of H^n. Since λ is conserved, trajectories of the system lie in level sets of λ. In the case that H^0 has an elliptic equilibrium at the origin, these level sets foliate the phase space, close to the origin, by compact sets that are homeomorphic to 3-spheres in $\mathbb{R}^4$. After dividing out the $\mathbb{S}^1$ symmetry, on a leaf with λ nonzero and fixed, we get an $\mathbb{S}^2$ (see 2.3.3). We go to a planar system by flattening out this $\mathbb{S}^2$ to a disk with boundary $\mathbb{D}^2$. The boundary is the image of an $\mathbb{S}^1$-orbit which is singular with respect to the reduction, and is called the *singular circle*. It coincides with

a singularity in the coordinate transformation to the plane. The reduced system so obtained is denoted by H^r.

The diameter of the singular circle depends on λ, and vanishes when λ does. If the planar coordinates are denoted x, y, then in x, y, λ-space the singular circles form a cone. From here on we ignore these singular circles, and consider the system in a full neighborhood of the origin in $\mathbb{R}^2 \times \mathbb{R}$. The system is most singular when $\lambda = 0$; this is called the *central singularity*, or *organizing center* [Mon94]. It is subjected to a transformation in order to bring it into a simple form. For the case of the spring-pendulum considered in Sect. 2.3, this is the $\mathbb{Z}_2$-symmetric hyperbolic umbilic. We are left with a deformation H^c of this singularity, in terms of the parameters b_1 and λ.

There exists a versal deformation of the hyperbolic umbilic with only two parameters. (In the non-equivariant case one finds three.) This deformation is denoted by H^u. In Sect. 2.2.6 we find the reparametrizations that induce H^c from H^u. This step is computationally involved, and is dealt with in the last two chapters. In this step we employ a standard basis and the corresponding division algorithm to compute the required morphisms efficiently.

Finally, we use the reparametrizations of Sect. 2.2.6 to compute the BCKV-restricted normal form H^B of our system.

2.2.2 Some notation

Symmetries and coordinate systems In the sequel, we use Cartesian canonical coordinates x_i, y_i as well as complex variables $z_i, \bar{z}_i$ and Hamiltonian polar coordinates L_i, ϕ_i, because certain transformations take a simple form in one of these coordinates. Two $\mathbb{Z}_2$-symmetries will also play a role: a mirror symmetry S which acts on the coordinates x_2, y_2 only, and a time-reversal symmetry T that acts on the momentum coordinates y_i. The relations between the coordinates and symmetries are as follows:

$$
\begin{array}{llll}
z_i = x_i + \mathrm{i}y_i = & \sqrt{2L_i}\mathrm{e}^{\mathrm{i}\phi_i}, & \bar{z}_i = x_i - \mathrm{i}y_i = & \sqrt{2L_i}\mathrm{e}^{-\mathrm{i}\phi_i} \\
\phi_i = \frac{1}{2\mathrm{i}} \log \frac{z_i}{\bar{z}_i} = & \arctan \frac{y_i}{x_i}, & L_i = \frac{1}{2} z_i \bar{z}_i = & \frac{1}{2}\left(x_i^2 + y_i^2\right) \\
x_i = \sqrt{2L_i}\cos\phi_i = & \frac{1}{2}(z_i + \bar{z}_i), & y_i = \sqrt{2L_i}\sin\phi_i = & \frac{1}{2i}(z_i - \bar{z}_i)
\end{array}
\tag{2.1}
$$

T:	S:
$(x_1, x_2, y_1, y_2) \mapsto (x_1, x_2, -y_1, -y_2)$	$(x_1, x_2, y_1, y_2) \mapsto (x_1, -x_2, y_1, -y_2)$
$(z_1, \bar{z}_1, z_2, \bar{z}_2) \mapsto (\bar{z}_1, z_1, \bar{z}_2, z_2)$	$(z_1, \bar{z}_1, z_2, \bar{z}_2) \mapsto (z_1, \bar{z}_1, -z_2, -\bar{z}_2)$
$(L_1, \phi_1, L_2, \phi_2) \mapsto (L_1, -\phi_1, L_2, -\phi_2)$	$(L_1, \phi_1, L_2, \phi_2) \mapsto (L_1, \phi_1, L_2, \phi_2 + \pi)$

Parameters and coefficients The dynamical systems we investigate depend on a number of variables. Certain variables are supposed to be constant during the evolution of the system, for example the mass of a pendulum. Throughout, we reserve the name *coefficient* for 'constant variables' that can take on arbitrary values, except possibly a few isolated ones excluded by non-degeneracy

conditions. The name *parameter* is reserved for 'constant variables' which are small. Asymptotic expansions are done in terms of phase space variables and parameters.

Hamiltonian contexts The Hamiltonian system H we consider appears in several versions, in the corresponding stages of the normalization process. The current 'stage' or context is denoted by a superscript, e.g. H^0 for the original Hamiltonian, H^n for the Birkhoff normal form.

Big-oh notation For brevity, we use the notation $O(|x, y|^k)$ to denote terms of total order k and higher in x and y. In standard notation, this would be $O((|x| + |y|)^k)$. Also, e.g. $O(|c_i, \lambda|^k)$ stands for $O((|c_1| + |c_2| + \cdots + |\lambda|)^k)$, when $c = (c_1, c_2, \ldots)$ is a vector of coefficients. This will be clear from the context.

Formal power series and functions In this work we often use formal power series. In order not to make the text unreadable, we shall all the same refer to them simply as *functions* or *maps* or *vector fields*. This is no problem since all operations on functions (maps, vector fields) are also allowed on formal power series, with the exception of conjugations with coordinate transformations that do not leave the origin fixed.

2.2.3 Birkhoff normalization

The first step in the reduction procedure is the application of the Birkhoff normal form. The quadratic part of the system's Hamiltonian H^0 determines the normal form. In particular, when the quadratic part is nondegenerate and nonresonant, the normal form system is integrable, and exhibits only trivial dynamics. On the other hand, if more than two harmonic oscillators are in resonance, reduction to 1 degree of freedom is not possible, and the planar reduction method cannot be used. We restrict our attention to the case of one resonance; for more remarks on this see the introduction to this chapter.

In Chap. 4 a description of the Birkhoff normal form procedure is given, together with algorithms that implement it. Here we give the results of the computation. The Birkhoff normal form of a Hamiltonian H^0 with an elliptic equilibrium at the origin, is determined chiefly by the kernel of ad_{H_2}, where H_2 is the quadratic part of H^0. Since the Birkhoff normal form procedure generally yields a divergent power series, we work in the ring $R = \mathbb{R}[[z_i, \bar{z}_i]]$ of formal power series (see Sect. 2.1 for remarks). Here $z_i, \bar{z}_i$ are complex coordinates that make ad_{H_2} act diagonally with respect to a monomial basis on R.

When H^0 is invariant under some symmetry group Γ (that respects the symplectic structure), the normal form procedure may be carried out within the ring of Γ-invariant power series R^Γ; see Chap. 4 remark 4.4. For the case of the symmetry groups considered above, we have the following result. (In general the problem of finding basic invariant polynomials for a given group action is difficult; see e.g. [Stu93].)

Proposition 2.1. *Let $H_2 = \mathrm{i}z_1\bar{z}_1 + \mathrm{i}\omega z_2\bar{z}_2$, where $\omega \neq 0$, and assume Γ is either $\{Id\}$, $\{Id, T\}$ or $\{Id, S, T, ST\}$. If $\omega \notin \mathbb{Q}$, a Hilbert basis for the algebra $\ker \mathrm{ad}_{H_2}$ in R^Γ is $\{z_1\bar{z}_1, z_2\bar{z}_2\}$. In the case that $\omega = p/q = P/Q$, where $p, P \neq 0$, $q, Q > 0$, Q even and $\gcd(p, q) = \gcd(P, Q/2) = 1$, the Hilbert basis for $\ker \mathrm{ad}_{H_2}$ additionally contains*

a) $\Gamma = \{Id\}$:	$z_1^p\bar{z}_2^q$, $\bar{z}_1^p z_2^q$ $(p \geq 0)$	or	$\bar{z}_1^{-p}\bar{z}_2^q$, $z_1^{-p}z_2^q$ $(p \leq 0)$
b) $\Gamma = \{Id, T\}$:	$z_1^p\bar{z}_2^q + \bar{z}_1^p z_2^q$ $(p \geq 0)$	or	$\bar{z}_1^{-p}\bar{z}_2^q + z_1^{-p}z_2^q$ $(p \leq 0)$
c) $\Gamma = \{Id, S, T, ST\}$:	$z_1^P\bar{z}_2^Q + \bar{z}_1^P z_2^Q$ $(p \geq 0)$	or	$\bar{z}_1^{-P}\bar{z}_2^q + z_1^{-P}z_2^q$ $(p \leq 0)$

A finite set of invariants generating the invariant ring of a Lie group is called a Hilbert basis; the terminology above is justified since $\ker \mathrm{ad}_{H_2}$ is the invariant ring of the Lie group consisting of exponentials of the vector field associated to H_2; see [Gat00, Hil93].

Note that since $S^2 = T^2 = Id$, the group Γ is isomorphic to $\mathbb{Z}_2$ in case (b), and isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ in case (c).

Applying Proposition 2.1 to the spring-pendulum system, which has 2 degrees of freedom, we get the following:

Proposition 2.2. *(Birkhoff normal form) Let H^0 be a Hamiltonian on $\mathbb{R}^4$ with vanishing linear part, invariant under T as defined in (2.1). Let $H_2^0 = \mathrm{i}z_1\bar{z}_1 + \mathrm{i}\omega z_2\bar{z}_2$ be its quadratic part, and assume that $\omega = \frac{P}{Q} = \frac{p}{q}$ with Q even, $Q, q > 0$ and $\gcd(P, Q/2) = \gcd(p, q) = 1$. Let*

$$\begin{aligned}
&\psi = z_1^p\bar{z}_2^q + \bar{z}_1^p z_2^q && \text{if } p > 0,\\
&\psi = z_1^{-p}z_2^q + \bar{z}_1^{-p}\bar{z}_2^q && \text{if } p < 0,\\
&\psi = z_1^P\bar{z}_2^Q + \bar{z}_1^P z_2^Q && \text{if } H^0 \text{ is } S\text{-invariant and } p > 0,\\
&\psi = z_1^{-P}z_2^Q + \bar{z}_1^{-P}\bar{z}_2^Q && \text{if } H^0 \text{ is } S\text{-invariant and } p < 0,
\end{aligned}$$

then there exists a formal symplectic T-equivariant coordinate transformation ϕ such that

$$H^n := H^0 \circ \phi = H_2^0 + f_0(z_1\bar{z}_1, z_2\bar{z}_2, \psi),$$

where $f_0(\zeta_1, \zeta_2, \zeta_3) = \mathrm{i}\alpha\zeta_3 +$ quadratic and higher order terms, for some $\alpha \in \mathbb{R}$. If H^0 is S-invariant, ϕ can be chosen to commute with S too. The quadratic part H_2^0 is conserved under the flow of H^n, i.e., H^n is invariant under the $\mathbb{S}^1$-action $A_\xi : (z_1, z_2) \mapsto (\mathrm{e}^{q\mathrm{i}\xi}z_1, \mathrm{e}^{p\mathrm{i}\xi}z_2)$, where $\xi \in \mathbb{S}^1 \equiv \mathbb{R}/2\pi\mathbb{Z}$. This action is nondegenerate except on the axes $z_1 = 0$ and $z_2 = 0$ on which points have isotropy subgroup (stabilizer) $\mathbb{Z}_p \subseteq \mathbb{S}^1$ and $\mathbb{Z}_q \subseteq \mathbb{S}^1$ respectively.

Remark 2.3. *(Complex coordinates)* Both in Proposition 2.1 as in Proposition 2.2 we use complex coordinates $z_i, \bar{z}_i$, connected to Cartesian coordinates through (2.1). Whenever we use complex coordinates, we shall assume that the symplectic form is $\mathrm{d}z \wedge \mathrm{d}\bar{z}$. The transformation from x_i, y_i to complex coordinates (with this symplectic form) is symplectic with *multiplier* 2i. Hence, real Hamiltonians in Cartesian (or Hamiltonian polar) coordinates correspond to purely imaginary Hamiltonians in complex coordinates. This explains why H_2^0 is purely imaginary in the propositions above.

2.2.4 Reduction to planar 1 degree-of-freedom system

We now discuss the reduction to the plane, for a system in $p : q$ resonance. The normalized system H^n has an additional (formal) $\mathbb{S}^1$ symmetry, with action $(z_1, z_2) \mapsto (e^{iq\xi} z_1, e^{ip\xi} z_2)$ for $\xi \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, and corresponding conserved quantity $H_2^0 = z_1\bar{z}_1 + \omega z_2\bar{z}_2$. This symmetry enables us to formally reduce to a one degree-of-freedom system. We first express the normalized system in Hamiltonian polar coordinates L_i, ϕ_i (see (2.1)):

$$H^n(L, \phi) = L_1 + \omega L_2 + f_1\left(L_1, L_2, L_1^{P/2} L_2^{Q/2} \cos(P\phi_1 - Q\phi_2)\right). \tag{2.2}$$

Here, and elsewhere in this section, the functions f_i are of the same form as f_0 in Proposition 2.2, differing only by innocent linear changes of variables. Let $p = P/\gcd(P, Q)$, $q = Q/\gcd(P, Q)$, and let r, s be integers such that $pr - qs = 1$. Consider the following symplectic coordinate change:

$$\begin{pmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{pmatrix} = \begin{pmatrix} r & s \\ q & p \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, \qquad \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix} = \begin{pmatrix} p & -q \\ -s & r \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{2.3}$$

The action of a symmetry group Γ will also have to be transformed to the new variables. In these variables, the system, the acquired $\mathbb{S}^1$-action and the action of the symmetry group generators S and T that will be used in the sequel, take the following form:

$$H^n(\tilde{L}, \tilde{\phi}) = \frac{1}{q}\tilde{L}_2 + f_2(\tilde{L}_1, \tilde{L}_2, (p\tilde{L}_1 - s\tilde{L}_2)^{P/2}(-q\tilde{L}_1 + r\tilde{L}_2)^{Q/2} \cos(\gcd(P, Q)\tilde{\phi}_1)),$$
$$T : \ (\tilde{\phi}_1, \tilde{\phi}_2) \mapsto (-\tilde{\phi}_1, -\tilde{\phi}_2),$$
$$S : \ (\tilde{\phi}_1, \tilde{\phi}_2) \mapsto (\tilde{\phi}_1 + q\pi, \tilde{\phi}_2 + r\pi),$$
$$\mathbb{S}^1\text{-action} : (\tilde{\phi}_1, \tilde{\phi}_2) \mapsto (\tilde{\phi}_1, \tilde{\phi}_2 + \xi),$$

from which it is manifest that $\tilde{L}_2$ is conserved (since the conjugate variable $\tilde{\phi}_2$ is cyclic), indeed, $\tilde{L}_2 = qH_2^0$. Note that the transformation (2.3) is invertible; in particular, the $\mathbb{S}^1$-action is nondegenerate (except at certain points). This may be contrasted to the q-sheeted cover used in [BV92]. We now reduce to a planar system by dividing out the $\mathbb{S}^1$-symmetry generated by $\tilde{L}_2$, viewing $\tilde{L}_2$ as a *distinguished parameter* We denote the planar reduction of H^n we get in this way by H^r. Calling $\tilde{L}_2$ a parameter is justified if we consider only small deviations from the system's lower equilibrium [1] for then the system has little energy, so H^n, and therefore $\tilde{L}_2$, is small. (See also remark 2.6.) To emphasize the role of $\tilde{L}_2$ as parameter, we write

$$\lambda := \tilde{L}_2 = qL_1 + pL_2$$

[1] i.e., the pendulum not moving and hanging straight down, with gravity balancing the spring force.

from now on. Next, we apply the translation $\bar{L}_1 = \tilde{L}_1 - \frac{s}{p}\lambda$, $\bar{\phi}_1 = \tilde{\phi}_1$, which in the context of planar Hamiltonian systems is a symplectic transformation. The Hamiltonian then becomes $H^r = \frac{\lambda}{q} + f_3(\bar{L}_1, \lambda, \bar{L}_1^{P/2}(\bar{L}_1 - \frac{\lambda}{pq})^{Q/2}\cos(\gcd(P,Q)\bar{\phi}_1))$. Finally, we return to Cartesian coordinates. Dropping the constant and hence dynamically irrelevant term λ/q, we get the following:

Proposition 2.4. *Under the assumptions of Proposition 2.2, let H^n be a Hamiltonian in Birkhoff normal form. There exist coordinates x, y, λ, ϕ on $\mathbb{R}^4$ such that λ is constant on orbits of H^n, and the projections of those orbits onto the (x,y)-plane coincide with those of a planar Hamiltonian system $H^r(x,y)$, with parameter λ and independent of ϕ, of the form*

$$P \text{ even:} \quad H^r = f_4\left(x^2+y^2\ ,\ \lambda\ ,\ (x^2+y^2)^{\frac{P}{2}-1}(x^2-y^2)\left(x^2+y^2-\frac{2\lambda}{pq}\right)^{Q/2}\right);$$

$$P \text{ odd:} \quad H^r = f_4\left(x^2+y^2\ ,\ \lambda\ ,\ (x^2+y^2)^{\frac{P-1}{2}}x\left(x^2+y^2-\frac{2\lambda}{pq}\right)^{Q/2}\right),$$

where $f_4(\zeta_1,\zeta_2,\zeta_3) = b_1\zeta_3 + h.o.t.$

Remark 2.5. *(Singular circle)* The coordinate transformation to Hamiltonian polar coordinates used in (2.2) is singular at the coordinate axes $L_1 = 0$ and $L_2 = 0$. These axes become $p\tilde{L}_1 - s\tilde{L}_2 = 0$ and $-q\tilde{L}_1 + r\tilde{L}_2 = 0$ in the transformed coordinates, and after translation $\bar{L}_1 = 0$ and $\bar{L}_1 = \lambda/pq$. The first singularity is removed by returning to Cartesian coordinates in the plane. The second singularity is called the *singular circle* (see Sect. 2.3.3). At this circle $L_2 = 0$ so that the coordinate ϕ_2 is ill-defined, and therefore so is $\bar{\phi}_1 = p\phi_1 - q\phi_2$. In particular this implies that H^r is constant on the circle; see also Sect. 2.3.3.

Remark 2.6. *(The parameter λ)* The adjective *distinguished* refers to the fact that λ stems from the phase space of H^n, and is a parameter only for the reduced system, not for the original one. If we are interested in the geometry of the local level sets of H^n on the full 4-dimensional phase space, we may not let reparametrizations of ordinary parameters depend on the distinguished parameter, see Sect. 2.2.7. This should be contrasted to the point of view taken in Sect. 2.2.6, where we classify the geometry of level sets in $\mathbb{R}^2$, and where it is permissible to treat λ as an ordinary parameter. Note that in either setting we do not allow reparametrizations of λ to depend on phase variables.

Remark 2.7. *(Symmetries)* When q is even, the acquired $\mathbb{S}^1$ normal form symmetry group contains the reflection $\mathbb{Z}_2$ symmetry S as a subgroup. Before reduction the symmetry group is therefore $\mathbb{S}^1 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{S}^1 \times \mathbb{Z}_2$, depending on the parity of q, leading to a symmetry group $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ for the reduced system. See also Proposition 2.1.

2.2.5 Reduction to the central singularity

At this point the system is reduced to a planar Hamiltonian H^r depending on a distinguished parameter λ and several ordinary parameters. Recall that parameters are supposed to be *small*. We now look at the 'degenerate' Hamiltonian that results when λ and the other parameters vanish. This is called the *central singularity*, also known as the *organizing center* [FSS98, Mon94] and is denoted by H_0^r.

In the cases we consider H_0^r is *finitely determined*, i.e., a finite piece of the jet of H_0^r is equivalent to H_0^r itself, via a smooth planar coordinate transformation ϕ. This transformation is independent of the parameters of H^r. Existence of this transformation is guaranteed by singularity theory. Since H^r is invariant under the action of a symmetry group, this has to be taken into account by singularity theory; see section 5.3 for details. The result is a polynomial normal form $H_0^r \circ \phi$ of H_0^r.

The *central singularity reduced normal form* is defined as $H^c := H^r \circ \phi$. At the origin of parameter space this is equal to the polynomial normal form, i.e., H^c is in fact a deformation of this polynomial normal form. The final step in the reduction process is to find a versal deformation of this normal form which can serve as a model for H^c. This versal deformation is denoted by H^u and forms the final model of the planar system.

2.2.6 Inducing the system from a versal deformation

With the versal deformation H^u in hand, we can find bifurcation curves in terms of the model's parameters. In order to pull-back these curves to the original parameter space, we need the explicit transformations that induce H^c from the versal deformation H^u. These transformations are used in the final step where we find the BCKV normal form, which incorporates the distinguished nature of the parameter λ.

Let us denote the small *parameters* in H^c by λ and $c_1, c_2, \ldots$, and the *coefficients* by $d_1, d_2, \ldots$. For the moment we disregard the distinguished nature of λ, treating it, like the c_i, as an ordinary parameter (see remark 2.6), and for notational convenience we write $\lambda = c_0$.

We want to find transformations that induce H^c from the versal deformation H^u. Assume that $H^u(x, y, u_1, \ldots, u_k)$ is a versal deformation of the central singularity; see [BL75, Mon91]. It follows that there exists a pair of transformations (ϕ, ρ), where $\phi : \mathbb{R}^2 \times \mathbb{R}^c \times \mathbb{R}^d \to \mathbb{R}^2$ is a parameter-dependent coordinate transformation, and $\rho : \mathbb{R}^c \times \mathbb{R}^d \to \mathbb{R}^k$ is a reparametrization from (c_i, d_j) to $(u_1, \ldots, u_k)$, such that

$$H^u(\phi(x, y, c_i, d_i), \rho(c_i, d_i)) = H^c(x, y, c_i, d_i).$$

These transformations obey the following additional constraints: ϕ is equivariant under the symmetry group, and both ϕ and ρ are trivial at the central singularity,

i.e., $\phi(x, y, 0, d_i) = (x, y)$ and $\rho(0, d_i) = 0$. In Sect. 5.3.1 we give a necessary and sufficient condition for a deformation to be versal. It amounts to solvability of the well-known *infinitesimal stability* equation[2] adapted to our equivariant context.

In the case of the 1 : 2 resonance, the central singularity is isomorphic to $x(x^2 + y^2)$, with a symmetry group $\mathbb{Z}_2$ acting on $\mathbb{R}^2$ via $(x, y) \mapsto (x, -y)$. A versal unfolding is $H^u = x(x^2 + y^2) + u_1 x + u_2 y^2$, and the condition boils down to: For every $\mathbb{Z}_2$-invariant germ g vanishing at the origin there should exist $\mathbb{Z}_2$-invariant germs $\alpha_i(x, y)$, $i = 1, 2, 3$ and real numbers u_1, u_2 such that

$$g(x,y) \;=\; \alpha_1(x,y)x\frac{\partial f}{\partial x} + \alpha_2(x,y)y^2\frac{\partial f}{\partial x} + \alpha_3(x,y)y\frac{\partial f}{\partial y} + u_1 x + u_2 y^2. \tag{2.4}$$

Here $f = x(x^2 + y^2)$ is the central singularity. For this f the condition is indeed satisfied; see Sect. 5.3.2.

Starting from the infinitesimal stability condition, versality is proved by invoking the Mather-Malgrange preparation theorem [Mar82, Poè76]. Our interest is not so much in proving existence, as in explicitly computing the transformations ϕ and ρ, up to a certain degree. An algorithm due to Kas and Schlessinger [KS72] accomplishes this; see Sect. 7.2.2. By using the fact that equations of the form (2.4) can be solved, it uses the solutions α_i and u_i to iteratively build the transformations ϕ and ρ. This algorithm can be regarded as a constructive proof of the existence of a *formal* solution for ϕ and ρ.

Our ability to compute ϕ and ρ now rests on our ability to compute solutions to (2.4). This can be done efficiently using standard bases; see Chap. 6 and Sect. 7.2.3.

2.2.7 BCKV normal form

BCKV theory classifies the family of systems H^r as 2 degree-of-freedom systems. For a given member of the family (i.e., for certain values of the coefficients) it provides a normal form system, which is itself a 2 degree-of-freedom system. This should be contrasted to the deformation H^u, which classifies H^r as a family of *planar* systems; see remark 2.6. The parameter λ is a phase-space variable that is constant under flows of the system. In the BCKV normal form, reparametrizations are not allowed to depend on λ.

It turns out to be possible to use H^u for constructing a suitable unfolding H^B (see theorem 5.16), corresponding to a generic path (surface) in a more general parameter space. (See also [Mon94].) In this setting many more parameters are needed for versality. The path arises when the coefficients of such a versal normal form are expressed as functions of the available parameters. Moreover, these parameters will be expressed in the original (physical) constants of the system. This gives the natural set-up for the aforementioned perturbation problem.

[2] See [GG73]. Necessity of this condition is immediate by considering deformations of the form $H^u(x, y, 0) + c_1 g(x, y)$ for arbitrary (symmetric) g; see [Mar82, Prop. IV.3.2].

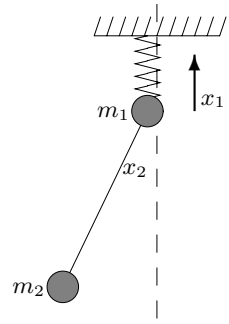

Fig. 2.1 The spring-pendulum with its axis of symmetry

See [BCKV93] for a general discussion. For the definition of BCKV-restricted unfoldings, see section 5.3.3.

2.3 Spring-pendulum in 1:2-resonance

This chapter ends with the application of the planar reduction method to the spring-pendulum system. We start by introducing the system, followed by the reduction to the polynomial normal form. At the end of this section we discuss the 1 : 2-resonant dynamics of the spring-pendulum, and give bifurcation diagrams, numeric Poincaré sections and a comparisons of the predicted pulled-back bifurcation curves with numerical data.

2.3.1 The system

The spring-pendulum is a planar pendulum suspended by a spring constrained to move along the vertical axis. It is a typical two degree-of-freedom Hamiltonian system with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ (time-reversal and reflection) symmetry.

We now describe the system. Masses are attached to both ends of the rod, while both rod and spring are massless. The configuration is given by the displacement of the suspension point and the angle of the pendulum with the vertical axis, denoted by x_1 and x_2. The potential energy is $U(x_1, x_2) = -m_2 gl \cos x_2 + \frac{1}{2}a^2 x_1^2$ when the origin is suitably chosen. The m_i denote masses, $M^2 = m_1 + m_2$, g the gravitational acceleration, l the length of the pendulum and a^2 the spring coefficient. The Hamiltonian of the system, expressed in configuration coordinates x_i and their conjugates y_i reads

$$H(x,y) = \tfrac{1}{2}a^2 x_1^2 - m_2 gl \cos x_2 + \frac{l^2 m_2 y_1^2 + M^2 y_2^2 - 2lm_2 y_1 y_2 \sin x_2}{m_2 l^2 (2M^2 + m_2[\cos(2x_2) - 1])}. \tag{2.5}$$

This Hamiltonian exhibits two $\mathbb{Z}_2$-symmetries, i.e., we have a symmetry group $\Gamma := \mathbb{Z}_2 \times \mathbb{Z}_2$. Generators are a time-reversible symmetry denoted by T, and reflection symmetry in the vertical axis, denoted by S; here

$$
\begin{aligned}
T &: (x_1, x_2, y_1, y_2) \mapsto (x_1, x_2, -y_1, -y_2), \\
S &: (x_1, x_2, y_1, y_2) \mapsto (x_1, -x_2, y_1, -y_2).
\end{aligned} \tag{2.6}
$$

Writing H as a Taylor series in the x_i and y_i variables, and applying a rescaling of variables and time to tidy up the quadratic terms, we get:

Proposition 2.8. *Provided that $m_2 \neq 0$ and $a \neq 0$, by a rescaling of variables and time we can bring the Hamiltonian (2.5) into the form*

(2.7)
$$
\begin{aligned}
H^0(x,y) := \frac{x_1^2+y_1^2}{2} + a_1\frac{x_2^2+y_2^2}{2} - 8a_2x_2y_1y_2 - 16a_3x_2^4 + 16a_4x_2^2y_1^2 + 16a_5x_2^2y_2^2 + \\
+ 32a_6x_2^3y_1y_2 + 64a_7x_2^6 + 64a_8x_2^4y_1^2 + 64a_9x_2^4y_2^2 + h.o.t.,
\end{aligned}
$$

where the h.o.t. are $O(|x,y|^7)$ terms, and the symplectic form is $\mathrm{d}x \wedge \mathrm{d}y$. Here $a_1 = \frac{\sqrt{g}M}{a\sqrt{l}}$ and $a_2 = \frac{1}{8al}$, and H^0 is invariant under S and T.

We use (2.7) as starting-point, with no conditions on the coefficients a_i. That is, we forget about the algebraic relations between the a_i that exist for this particular system. The system (2.7) has the same qualitative form as the spring-pendulum system. In fact, for a proper choice of the coefficients a_i, and modulo a rescaling, the latter is a high order perturbation of (2.7).

The physical origin of the system imposes some constraints on the coefficients, for example $a_1 > 0$ and $a_2 > 0$. We will not use these. Instead, we keep an eye on the non-degeneracy conditions encountered during the calculations, allowing the a_i to otherwise take arbitrary values. It turns out that some of these conditions are implied by the physical constraints.

2.3.2 Reduction

Birkhoff normalization, reduction to a planar system Let us denote by H_2 the quadratic part of H^0. At $a_1 = \frac{1}{2}$, the kernel of ad_{H_2} is generated as an algebra by $z_1\bar{z}_1$, $z_2\bar{z}_2$ and $z_1\bar{z}_2^2 + \bar{z}_1 z_2^2$. These generators are invariant under the action of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry generated by S and T. After Birkhoff normalizing the spring-pendulum Hamiltonian can be written as

$$
H^n = H_2 + f(z_1\bar{z}_1, z_2\bar{z}_2, z_1\bar{z}_2^2 + \bar{z}_1 z_2^2).
$$

In the new Birkhoff coordinates, the quadratic part of the Hamiltonian is an integral of motion. We denote this integral by $\lambda := 2H_2$, see also (2.9) below. Later on, we refer to this integral as the *distinguished parameter*. On the original phase space it has the expression

(2.8)
$$\begin{aligned}\lambda = {} & \left(x_1^2+y_1^2\right) + \frac{1}{2}\left(x_2^2+y_2^2\right) + \frac{8a_2}{1+2a_1}\left(x_1x_2^2 - 2x_2y_1y_2 - x_1y_2^2\right)\\ & + \frac{4a_4}{a_1(1-a_1^2)}\left((1-4a_1+2a_1^3)x_1^2x_2^2 + (1+4a_1-2a_1^2-2a_1^3)x_2^2y_1^2 + \right.\\ & \qquad \left.(4a_1-8a_1^2)x_1x_2y_1y_2 + (-1+2a_1^3)x_1^2y_2^2 + (-1+2a_1^2-2a_1^3)y_1^2y_2^2\right)\\ & + \frac{64a_2^2}{(1+2a_1)^2}\left((1+a_1)x_1^2 - a_1y_1^2\right)\left(x_2^2+y_2^2\right)\\ & + \frac{4a_2^2}{a_1(1+2a_1)^2}\left(-2(3+4a_1)x_2^2y_2^2 + (1+4a_1)(x_2^4+y_2^4)\right)\\ & + \frac{(-10a_3-2a_5)x_2^4 + 12(a_3+a_5)x_2^2y_2^2 + (6a_3-2a_5)y_2^4}{a_1} + O(|x_i,y_i|^5).\end{aligned}$$

We now transform to Hamiltonian polar coordinates. The generators $z_i\bar{z}_i$ and $z_1\bar{z}_2^2 + \bar{z}_1z_2^2$ take the form L_1, L_2 and $\sqrt{L_1}L_2\cos(\phi_1 - 2\phi_2)$. The coordinates L_1, L_2 are globally defined, and the angles ϕ_1, ϕ_2 are well-defined for $L_1 \neq 0$ and $L_2 \neq 0$, respectively. To reduce to a planar system we apply the symplectic transformation in these coordinates (see (2.3)),

$$\bar{L}_1 = L_1, \quad \lambda = 2L_1 + L_2, \quad \bar{\phi}_1 = \phi_1 - 2\phi_2, \quad \bar{\phi}_2 = \phi_2. \tag{2.9}$$

This transformation is singular for $L_2 = 0$ and $L_1 \neq 0$, since then $\bar{L}_1 \neq 0$ whereas $\bar{\phi}_1$ is ill-defined. This happens at $\bar{L}_1 = \lambda/2$, and the corresponding circle (in the $\bar{L}_1, \bar{\phi}_1$ polar coordinates) is called the *singular circle*.

The normalized system is independent of $\bar{\phi}_2$, whereas the new coordinate $\bar{\phi}_1$ becomes constant in the unperturbed ($\omega = \frac{1}{2}$) linear flow. After this we return to complex coordinates by the usual transformation (2.1). In complex coordinates, transformation (2.9) reads

$$z_1 = z_1'\frac{z_2'}{\bar{z}_2'}, \quad z_2 = z_2'\sqrt{1 - 2\frac{z_1'\bar{z}_1'}{z_2'\bar{z}_2'}}, \tag{2.10}$$

where z_i are the old complex coordinates. The singular circle in new coordinates is $z_1'\bar{z}_1' = \frac{1}{2}z_2'\bar{z}_2'$ (a codimension-1 subspace; it is a circle only after reduction).

Proposition 2.9. *(Planar reduction) After Birkhoff normalization and reduction to one degree of freedom, for the* $1:2$ *resonance* ($a_1 \approx \frac{1}{2}$)*, the Hamiltonian (2.7) takes the form*

$$\begin{aligned}H^r = {} & b_1\zeta_1 + b_2\zeta_2 + \frac{1}{b_3^3}\zeta_3 + b_4\zeta_1^2 + b_5\zeta_1\zeta_2 + b_6\zeta_2^2 + \\ & b_7\zeta_1\zeta_3 + b_8\zeta_2\zeta_3 + b_9\zeta_1^3 + b_{10}\zeta_1^2\zeta_2 + b_{11}\zeta_1\zeta_2^2 + b_{12}\zeta_2^3 + b_{13}\zeta_3^2 + O(|z_i,\bar{z}_i|^y),\end{aligned}$$

where $\zeta_1 = x^2+y^2$, $\zeta_2 = \lambda$, $\zeta_3 = x(x^2+y^2-\lambda)$, *and the coefficients for the terms up to order four in the original phase coordinates are given by*

$$b_1 = \frac{1}{2} - a_1; \quad b_2 = a_1; \quad b_3 = \frac{1}{\sqrt[3]{a_2}};$$
$$b_4 = 8\left(\frac{a_2^2}{1+2a_1} - 3a_3 - a_4 + a_5\right); \quad b_5 = 8(6a_3 + a_4 - 2a_5);$$
$$b_6 = -8\left(\frac{a_2^2}{1+2a_1} + 3a_3 - a_5\right).$$

The special form for the coefficient of ζ_3 was chosen for notational convenience in the formulas below. The coefficient b_1 vanishes at resonance ($a_1 = \frac{1}{2}$). It is considered to be small throughout, and is referred to as *detuning parameter*, since it measures the deviation from the resonant frequency.

Remark 2.10. *(Nondegeneracy conditions)* From the expression of the b_i, the first nondegeneracy condition can be read off: $1 + 2a_1 \neq 0$. If we continue to normalize to higher orders, more conditions of the form $a_1 \neq p/q$ are found, where $p/q \in \mathbb{Q}$.

Reduction to the central singularity Because the system is planar now, we may use general ($\mathbb{Z}_2$-equivariant) planar transformation ϕ for further normalization, as opposed to just the symplectic ones. The resulting system is not dynamically conjugate but *equivalent* to the original, i.e., it is conjugate modulo state-dependent time-reparametrizations; for more remarks see [BCKV95, BHLV98, BLV98, BCKV93].

The central singularity is defined by $b_1 = 0$ (resonance) and $\lambda = 0$. At this point the singularity still depends on the coefficients $b_2, b_3, \ldots$. In this section we bring the system at the central singularity in polynomial normal form $x(x^2+y^2)$, which is independent of the b_i. This singularity is the $\mathbb{Z}_2$-invariant hyperbolic umbilic (see [PS78]), in Arnol'd's classification denoted by D_4^+.

First, by a simple scaling transformation ϕ_0 we can achieve that the Hamiltonian takes the form $H^{r\prime} := H^r_{b_1=\lambda=0} \circ \phi_0 = x(x^2+y^2) + h.o.t.$

Remark 2.11. *(Nondegeneracy conditions)* This is possible provided that the coefficient of the third-order terms (in x, y) are nonzero. This translates into the condition $a_2 \neq 0$ (see Proposition 2.9).

Next, we look for a near-identity planar morphism ϕ removing the $h.o.t.$ from $H^{r\prime}$. This morphism should respect the $\mathbb{Z}_2$ symmetry $(x, y) \mapsto (x, -y)$. By a generalization of [Mar82, theorem III.5.2] incorporating the symmetry group, $H^{r\prime}$ is isomorphic to $x(x^2+y^2)$; for details see Proposition 5.11 in Sect. 5.3.2.

Armed with the knowledge that ϕ exists we set out to compute it, using the following iterative approach. Set $\phi_1(x) = x$, and assume that

$$H^{r\prime} \circ \phi_k = x(x^2+y^2) + O(|x, y|^{k+3}) \tag{2.11}$$

for some k. To find $\phi_{k'}$ with $k' = k+1$ we set $\phi_{k'} = \phi_k + \sum \alpha_i t_i$, where $\{t_i\}$ span the space of $\mathbb{Z}_2$-equivariant terms in x, y of degree k'. This results in a

set of linear equations for the real numbers α_i. By existence of the normalizing transformation, this set of equations is not over-determined, in fact it is usually under-determined.

Proposition 2.12. *There exists a coordinate transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $H^c := H^r \circ \phi$ is of the form*

$$\begin{aligned} H^c = &(1+c_1)x(x^2+y^2) + c_2(x^2+y^2) + c_3x^4 + c_4x^2y^2 + c_5y^4 + \\ &\lambda(d_1x + d_2x^2 + d_3y^2 + d_4x^3 + d_5xy^2 + d_6x^4 + d_7x^2y^2 + d_8y^4) + \\ &\lambda^2(d_{12}x + d_{13}x^2 + d_{14}y^2) + h.o.t., \end{aligned}$$

where the h.o.t. are terms of order $O(|x,y|^5)$, $O(|c_i,\lambda|^3)$ and $O(|x,y|^3|c,\lambda|^2)$. Here $d_i = d_i(b_j)$ are coefficients, and $c_i = c_i(b_j)$ are parameters, all of them rational expressions in the b_i, with the c_i vanishing at $b_1 = 0$. Up to third order terms, the following is a suitable transformation ϕ:

$$\phi : \begin{cases} x \mapsto b_3x - \frac{1}{3}b_3^5b_4x^2 + \left(\frac{1}{3}b_3^9b_4^2 - \frac{1}{3}b_3^6b_7\right)x^3 - b_3^5b_4y^2 - \left(\frac{1}{3}b_3^9b_4^2 + b_3^6b_7\right)xy^2, \\ y \mapsto b_3y + \frac{2}{3}b_3^5b_4xy + \left(\frac{2}{3}b_3^9b_4^2 - \frac{4}{3}b_3^6b_7x^2y\right)x^2y. \end{cases}$$

Proof: This is a corollary to Proposition 5.11. The transformation ϕ was computed using the algorithm outlined above. ∎

We say that H^c is in *central singularity reduced form*, i.e., at the central singularity $b_1 = \lambda = 0$ it reduces to the normal form $x(x^2+y^2)$.

We consider ϕ in the above proposition to be *fixed*, i.e., independent of parameters. It does depend on the coefficients $b_2, b_3, \ldots$ however, since $H^r|_{\lambda=b_1=0}$ also depends on those. Some leading order parameters and coefficients of the deformation H^c are:

$$\begin{aligned} c_1 &= -\frac{2}{3}b_1b_3^6b_4, \quad c_2 = b_1b_3^2, \\ d_1 &= -\frac{1}{b_3^2}, \quad d_2 = \frac{1}{3}b_3^2(b_4+3b_5), \quad d_3 = b_3^2(b_4+b_5). \end{aligned}$$

Inducing the system from a versal deformation From proposition 5.12 in Chap. 5 it follows that $H^u = x(x^2+y^2) + u_1x + u_2y^2$ is a versal deformation of the hyperbolic umbilic $x(x^2+y^2)$ (in the context of $\mathbb{Z}_2$-symmetric potential functions). Our system H^c is now normalized to the extent that the algorithms of Sect. 7.2 may be applied. The result is as follows:

Proposition 2.13. *Let H^c be a planar Hamiltonian depending on parameters c_i and coefficients d_i, with central singularity $x(x^2+y^2)$ at $c_0 = c_1 = \ldots = 0$, symmetric under the $\mathbb{Z}_2$-action $(x,y) \mapsto (x,-y)$. A versal deformation of this central singularity is given by*

$$H^u := x(x^2+y^2) + u_1x + u_2y^2,$$

so that there exist ϕ and ρ such that

$$H^c = H^u(\phi(x, y, c_i, d_i), \rho_1(c_i, d_i), \rho_2(c_i, d_i)), \tag{2.12}$$

with $\phi(x, y, 0, d_i) = (x, y)$, $\rho(0, d_i) = (0, 0)$. To compute

a) ϕ modulo $O(|x, y|^A)+O(|c_i|^B)$, it is sufficient to know H^c modulo $O(|x, y|^{A+2})+O(|c_i|^B)$;

b) ρ modulo $O(|c_i|^B)$, it is sufficient to know H^c modulo $O(|c_i|^B) + O(|x, y|^3)$.

For system H^c of Proposition 2.12, modulo $O(|c_i, \lambda|^3)$ terms, and writing λ instead of c_0 again, the reparametrization ρ reads

$$\begin{aligned}
u_1 &= \left(-\frac{1}{3}c_2^2 + O(c_i^3)\right) + \lambda\left(d_1 - \frac{1}{3}c_1 d_1 - \frac{2}{3}c_2 d_2 + O(c_i^2)\right) + \\
&\quad \lambda^2\left(d_{12} - \frac{1}{3}d_2^2 - \frac{1}{3}d_1 d_4 + O(c_i)\right) + O(\lambda^3), \\
u_2 &= \left(\frac{2}{3}c_2 - \frac{4}{9}c_2 c_1 + O(c_i^3)\right) + \\
&\quad \lambda\left(-\frac{1}{3}d_2 + d_3 + \frac{1}{9}c_3 d_1 - c_5 d_1 + \frac{2}{9}c_1 d_2 - \frac{2}{3}c_1 d_3 + \frac{5}{9}c_2 d_4 - c_2 d_5 + O(c_i^2)\right) + \\
&\quad \lambda^2\left(-\frac{1}{3}d_{13} + d_{14} + \frac{2}{9}d_2 d_4 + \frac{1}{3}d_3 d_4 - d_3 d_5 + \frac{1}{9}d_1 d_6 - d_1 d_8 + O(c_i)\right) + O(\lambda^3).
\end{aligned}$$

The coordinate transformation ϕ, modulo $O(|x, y|^3) + O(|c_i, \lambda|^2)$ terms, reads

$$\begin{aligned}
x &\mapsto \frac{1}{3}c_2 + \frac{1}{3}d_2\lambda + \left(1 + \frac{1}{3}c_1 + \frac{1}{3}d_4\lambda\right)x + \left(\frac{1}{3}c_3 + \frac{1}{3}d_6\lambda\right)x^2 + (c_5 + d_8\lambda)\,y^2 \\
y &\mapsto \left(1 - \frac{1}{6}d_4\lambda + \frac{1}{2}d_5\lambda + \frac{1}{3}c_1\right)y + \left(\frac{1}{2}c_4 - \frac{3}{2}c_5 - \frac{1}{6}c_3 + \frac{1}{2}d_7\lambda - \frac{3}{2}d_8\lambda - \frac{1}{6}d_6\lambda\right)xy.
\end{aligned}$$

Remark 2.14. *(Relevant degree for H^n)* To compute ρ up to second order, it suffices to know H^c modulo $O(|c_i, \lambda|^3) + O(|x, y|^3)$ terms. In turn, for this, H^n modulo $O(|x, y|^7)$ terms suffices, as λ is a quadratic polynomial on the phase space of H^n. To compute ϕ up to terms given in Proposition 2.13, it suffices to know H^c modulo $O(|x, y|^5) + O(|c_i, \lambda|^2)$ terms, and again H^n modulo $O(|x, y|^7)$ terms suffices.

Remark 2.15. *(Singular circle)* In Sect. 2.3.3 the singular circle of H^u is defined as the circular level set that touches the two saddle points arising for $u_1 < 0$ (see Fig. 2.4). By a topological argument, its pull-back by ϕ must coincide with the singular circle of H^c, defined as the set of singular points of (2.10). Up to the order in x, y, c_i and λ that ϕ and ρ were computed, we verified that they indeed do.

Proof (of proposition 2.13): The first part is proved by inspecting the Kas and Schlessinger's algorithm described in section 7.2.2 and 7.2.3, and the division algorithm 6.14 of Sect. 6.3.7. The fact that H^c is required up to order $A+2$ in order to compute ϕ only up to degree A is due to the first derivatives of the central singularity being of second degree. Similarly, in order to fix ρ, it is sufficient to compute H^c up to degree 2 in (x,y) as the deformation directions associated to ρ_1 and ρ_2 are of degree 2 or less (namely x and y^2 respectively). A little computer algebra yields the second part. ∎

BCKV normal form of H^c The constructive proofs of Proposition 5.19 and Lemma 5.18 provide an algorithm for computing the BCKV normal form. Using the reparametrizations of Proposition 2.13, we choose for Υ the following:

$$\Upsilon(\lambda, c_i) = (u_2(\lambda, c_i) - u_2(\lambda, 0), c_1, u_2(0, c_i), c_3, c_4, \ldots),$$

which is invertible, and then $\tilde{\sigma}_2 := u_1 \circ \Upsilon^{-1}$. The result is the following:

Theorem 2.16. *(BCKV normal form:) The system H^c of Proposition 2.12 is equivalent, modulo BCKV-restricted morphisms and reparametrizations, and modulo terms of order $O(|c_i, \lambda|^3)$, to*

$$\begin{aligned} H^B(x,y,\lambda,c_i) = {} & x(x^2+y^2) + y^2(\lambda + c_2) + x \times \left(-\frac{c_2^2}{3} + O(c_i^3) + \right. \\ & \left(d_1 - \frac{c_1 d_1}{3} - \frac{2c_2 d_2}{3} + O(c_i^2) \right) \beta\lambda + \left(d_{12} - \frac{d_2^2}{3} - \frac{d_1 d_4}{3} \right) \beta^2\lambda^2 + \\ & \left(\frac{d_1 d_{13}}{3} - d_1 d_{14} - \frac{2d_1 d_2 d_4}{9} - \frac{d_1 d_3 d_4}{3} + d_1 d_3 d_5 - \frac{d_1^2 d_6}{9} + d_1^2 d_8 + O(c_i) \right) \beta^3\lambda^2 \\ & \left. + O(\lambda^3) \right), \end{aligned}$$

where $\beta = 9(-3d_2 + 9d_3 + c_1(2d_2 - 6d_3) + c_2(5d_4 - 9d_5) + c_3 d_1 - 9c_5 d_1)^{-1} + O(c_i^2)$
The coefficient of x expressed in the a_i reads:

$$\begin{aligned} \tilde{\sigma}_2 = {} & -\frac{1}{48\alpha}(1-2a_1)^2 + O((1-2a_1)^3) + \\ & \lambda \left(\frac{-9a_2^2}{2\alpha\delta} + \frac{1}{4\alpha\delta^2} \left(2a_2^4 - 144a_3^2 + 5a_4^2 - 6a_3(a_4 - 16a_5) + 2a_4 a_5 + \right. \right. \\ & \left. \left. -16a_5^2 + a_2^2(-30a_3 + a_4 + 26a_5) + 3a_2 a_6 \right) (1-2a_1) + O((1-2a_1)^2) \right) \\ & + O(\lambda^2), \end{aligned}$$

where $\alpha = \sqrt[3]{2a_2^2}$ *and* $\delta = 2a_2^2 + 6a_3 - a_4 - 2a_5$.

Remark 2.17. *(Nondegeneracy conditions)* The BCKV normal form is only well-defined if β is, i.e., if $d_2 - 3d_3 \neq 0$. This translates into $a_2 \neq 0$ and $a_2^2 \neq (1+2a_1)(3a_3 + a_4 - a_5)$. For the spring-pendulum the first condition is trivial, the second one is not.

2.3.3 Dynamics and bifurcations

The planar system In this section we regard the system as a *planar* system depending on the detuning parameter $1-2a_1$ and distinguished parameter λ. This gives an integrable approximation to the dynamics of the iso-energetic, or equivalently[3] iso-λ, Poincaré map. In Sect. 2.2.6 we arrived at the planar versal normal form

$$H^u(x,y,u_1,u_2) = x(x^2+y^2) + u_1x + u_2y^2.$$

The level sets of this deformation are organized by a special level set that factorizes into first- and second-degree algebraic curves crossing in the point $(x,y) = (-u_2, \pm\sqrt{-u_1-3u_2^2})$. These curves are given by $x=-u_2$ and $(x-\frac{1}{2}u_2)^2+y^2 = -u_1-\frac{3}{4}u_2^2$. For parameter values for which these curves cross, the second equation defines a circle separating compact level curves from unbounded ones. This circle is referred to as the *singular circle*, the reason being that it is the image of singular points of the transformation (2.10).

The deformation has the critical points $(x,y) = (\pm\sqrt{-(1/3)u_1}, 0)$ and $(-u_2, \pm\sqrt{-u_1-3u_2^2}$. Saddle-center and and Hamiltonian pitchfork bifurcations occur along the curves $u_1=0$ and $u_1+3u_2^2=0$, respectively (see figure 2.4). Plugging in the reparametrizations found in Proposition 2.13, Sect. 2.3.2, yields implicit equations for these bifurcation curves in the $(\lambda, 1-2a_1)$-plane. For practical reasons we choose to solve for λ in terms of $1-2a_1$. The result is:

Proposition 2.18. *In the reduced system H^c of Proposition 2.12, saddle-center and Hamiltonian pitchfork bifurcations respectively occur along the following curves in parameter space:*

$$(u_1 = 0:) \qquad (2.13)$$
$$\lambda = -\frac{(1-2a_1)^2\left(4(5+8a_1)a_2^2 + (4a_1^2-1)(24a_3+5a_4-8a_5)\right)}{3456(1+2a_1)a_2^4} + \\ +O((1-2a_1)^3),$$
$$(u_1+3u_2^2 = 0:) \qquad (2.14)$$
$$\lambda = \frac{(1-2a_1)^2}{64a_2^2} + \frac{(a_2^2-a_4)(1-2a_1)^3}{128a_2^4} + O((1-2a_1)^4).$$

Remark 2.19. *(Phantom bifurcation)* The parameter λ is nonnegative, and close to resonance ($a_1 \approx \frac{1}{2}$) the solution (2.13) is negative. In the system H^0, therefore, the corresponding bifurcation does not occur. This conclusion also follows from the observation that at the bifurcation (2.13) the singular circle disappears, whereas H^r exhibits this singularity for all parameter values (as long as $\lambda > 0$).

The second solution does define a bifurcation, however. We continue with its description.

[3] see [BCKV93, Sect. 4.2]

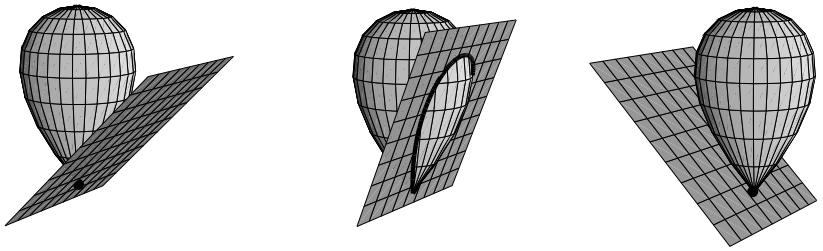

Fig. 2.2 Qualitative picture of intersections of the reduced phase space with level sets of H^n (here depicted as planes, which is the limiting situation for small energies, i.e., small balloons) through the singular point in ρ_2, ψ, χ-space, for three values of the detuning parameter.

Bifurcations and dynamical implications First we discuss the bifurcation of the reduced system H^c in the plane. If we let a_1 deviate sufficiently far from the resonant value $\frac{1}{2}$, the corresponding points in the (u_1, u_2)-plane in Fig. 2.4 will trace out a line that crosses the parabola twice, as u_1 is always negative.

Assume the parabola is crossed from below. Then at first the system has one maximum inside the singular circle, and a saddle point outside it. After the first Hamiltonian pitchfork bifurcation, two saddle points have formed on the singular circle, together with a minimum inside, with no critical points outside. The two saddle points have a heteroclinic connection because of the $\mathbb{Z}_2$ symmetry.

The second bifurcation destroys the maximum, leaving only a minimum inside the circle, and again a saddle outside of it.

Topological remarks *A priori* the spring-pendulum lives on the fixed-energy submanifold in $\mathbb{R}^4$, in our case $\mathbb{S}^3$. This sphere is homeomorphic to $\mathbb{D}^2 \times \mathbb{S}^1$ modulo an identification on $\partial\mathbb{D}^2 \times \mathbb{S}^1$.

The normalized Hamiltonian H^n on $\mathbb{S}^3$ has an $\mathbb{S}^1$-symmetry. There is one $\mathbb{S}^1$-orbit on which points have stabilizer (or isotropy subgroup) $\mathbb{Z}_2$, see Proposition 2.2; all other points have trivial stabilizer. As an illustration of this topology, consider the following model of $\mathbb{S}^3$ in the form of a map $\mathbb{D}^2 \times \mathbb{S}^1 \to \mathbb{S}^3$ given by

$$(x, y, \phi) \mapsto (\sqrt{1-r^2}\cos\phi, \sqrt{1-r^2}\sin\phi, x\cos 2\phi - y\sin 2\phi, x\sin 2\phi + y\cos 2\phi).$$

(Here $r^2 = x^2 + y^2$ and $\mathbb{D}^2 = \{r^2 \leq 1\}$.) This map is surjective, and injective on the interior of its domain. It provides a correspondence between $\mathbb{S}^1$-invariant functions on $\mathbb{S}^3$ and functions on $\mathbb{D}^2$ that are constant on $\partial\mathbb{D}^2$, that is, functions on $\mathbb{S}^2$. This conclusion holds for any[4] nondegenerate $\mathbb{S}^1$-action on $\mathbb{S}^3$, and justifies viewing the bifurcations described above on $\mathbb{S}^2$. This is the content of the remark

[4] There exist mutually non-homotopic, even non-homeomorphic, nondegenerate $\mathbb{S}^1$-actions on $\mathbb{S}^3$. This is in contrast to the case of $\mathbb{S}^2$: up to homotopy between actions there exists only 1 nondegenerate $\mathbb{S}^1$-action on $\mathbb{S}^2$. Here, a *nondegenerate* action is an action that maps nonidentity elements to nonidentity elements.

in Sect. 2.2, where we said that $\mathbb{S}^3$ divided out by an $\mathbb{S}^1$-action gives $\mathbb{S}^2$. After this division, the singular circle collapses to a single point on $\mathbb{S}^2$, and is referred to as the *pole*.

More precise information can be obtained by looking at the algebra. In general, the normalized Hamiltonian can be written as $H^n = f(\rho_1, \rho_2, \psi, \chi)$ where $\rho_1 = z_1\bar{z}_1$, $\rho_2 = z_2\bar{z}_2$, $\psi = z_1\bar{z}_2^2 + \bar{z}_1 z_2^2$ and $\chi = (z_1\bar{z}_2^2 - \bar{z}_1 z_2^2)/\mathrm{i}$ are invariants generating the $\mathbb{S}^1$-invariant functions (i.e., they form a Hilbert basis). Note that for time-reversible Hamiltonians the Birkhoff normal form is independent of χ, which explains why χ does not appear in Proposition 2.2. The invariants satisfy the relation $\rho_1\rho_2^2 = (\psi^2 + \chi^2)/4$. Moreover, reality conditions imply $\rho_1 \geq 0$ and $\rho_2 \geq 0$. The quadratic part H_2 is an integral of H^n, and without loss of generality we may reduce to $H_2 = 2\rho_1 + \rho_2 = \epsilon$, where ϵ is some small positive number. Then, the relation between the invariants defines a 2-dimensional manifold in the real space $\mathbb{R}^3 \ni (\rho_1, \psi, \chi)$, the *reduced phase space*, namely $(\epsilon - 2\rho_1)^2\rho_1 = \frac{1}{4}(\psi^2 + \chi^2)$. Topologically it is a sphere in this real space, but it has a cone-like singularity at $\rho_2 = 0$ (see figure 2.2). This singularity has dynamical significance: it is always a fixed point.

We now interpret the bifurcations on this (topological) sphere. Levels of H^n are surfaces in $\mathbb{R}^3 \ni (\rho_1, \psi, \chi)$ and intersect the reduced phase space in a curve; again, see Fig. 2.2. For small energies the level sets of H^n will be approximately planar on the scale of the reduced phase space ('balloon'), and the intersection curves are smooth circles, except for the level of the pole.

As in the previous section, suppose we traverse the (u_1, u_2)-plane on the left of the u_2-axis crossing the parabola of Hamiltonian pitchfork bifurcations twice. First, the Hamiltonian has one maximum somewhere on the sphere, and a minimum at the pole. The homoclinic[5] connection appearing in the planar normal form after the first bifurcation (see Fig. 2.5) corresponds to a level curve passing through the pole. In this situation, the pole is no longer a minimum. At the second bifurcation the homoclinic connection disappears, implying that the pole is an extremum again, now a maximum.

Dynamics of the spring-pendulum At the pole, $\rho_2 = 0$ and ρ_1 is a maximum, corresponding to the pendulum oscillating vertically without swinging ($x_2 = 0$, see figure 2.1). Points on this periodic orbit have nontrivial stabilizer, $\mathbb{Z}_2$; in other words, the period of this orbit is (to first order) half that of the other periodic orbits. In the literature it is referred to as the *short periodic orbit*. Since the pole always exists and is always a fixed point, the short periodic orbit always exists. Far from resonance (outside the parabola of Fig. 2.4) the pole is an extremum of the Hamiltonian, so that this orbit is stable. It is unstable close to resonance.

In that situation, the spring-pendulum exhibits two stable periodic trajectories (the *long periodic orbits*), corresponding to the two extrema existing on the sphere (see the center picture in Fig. 2.2). Physically, the lower mass traces

[5] The connection is not heteroclinic; see e.g. Fig. 2.2, center picture. In Fig. 2.5 the pole is blown up to a circle, so that there the connection *seems* to be heteroclinic.

out ∪-shaped and ∩-shaped paths, respectively. As the system moves away from resonance, one of these paths gets wider while the other gets narrower, until at bifurcation, the narrow one coincides with the short periodic orbit. After the bifurcation only one long periodic orbit remains (which is stable); the short periodic orbit has also become stable.

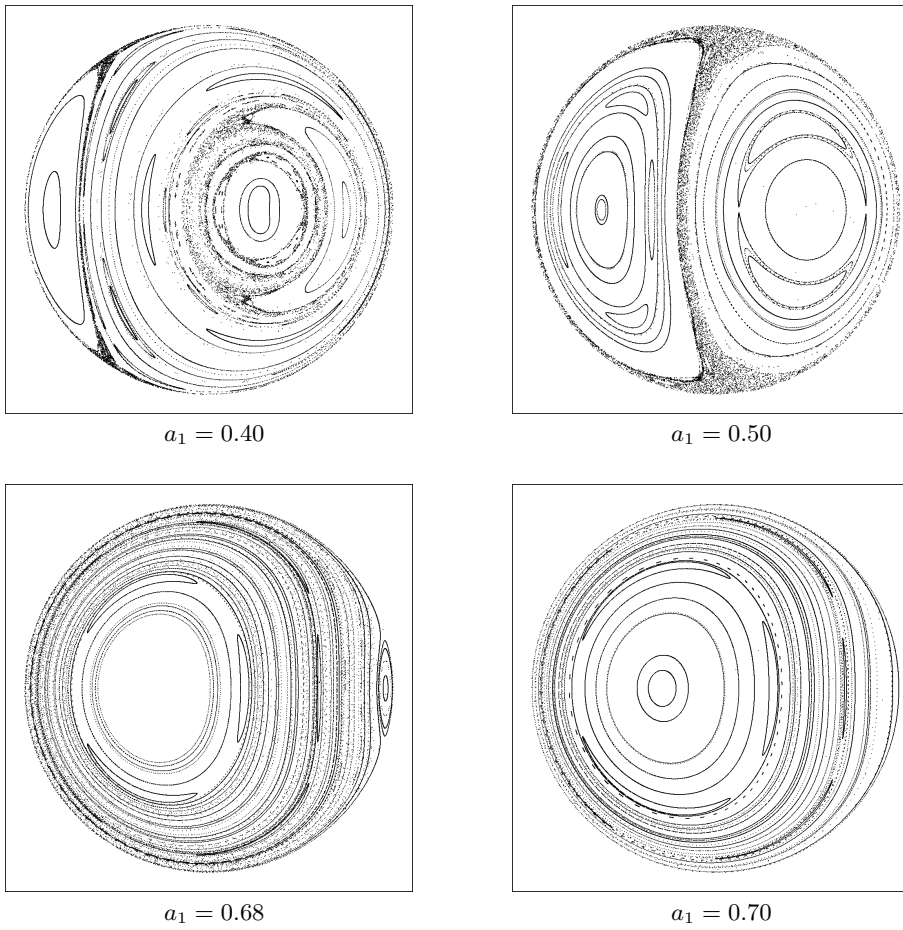

Fig. 2.3 Orbits of iso-energetic Poincaré map of H^0 near $1:2$ resonance, for various values of detuning parameter $1 - 2a_1$ (see Proposition 2.9). For these pictures we used $a_2 = 0.07$, $a_3 = 0.001$, other coefficients zero and $H^0 = 0.2$.

Comparison with numerical simulations To check the results above, we integrated H^0 numerically, and plotted the iso-energetic Poincaré section $\phi_2 = 0$

for varying values of the energy and detuning parameter a_1. The resulting pictures, shown in figure 2.3, are similar to those found by computation and shown in Fig. 2.5. The differences (chaotic regions, subharmonics) are caused by the flat perturbation between the normalized H^0 and H^n, destroying integrability in H^0; see also remarks in Sect. 2.1.

To check (2.14), we located some bifurcation points, by varying the detuning parameter a_1 for fixed H, a_2 and a_3. Other a_i were set to zero. The results are given in table 2.2. To compute λ we used equation (2.8); we see that for these values of the energy, $\lambda = 2H$ to good approximation. The final column gives the bifurcation value of λ given by (2.14) in each situation. The agreement with the measured value of λ is very good, especially for small H, as expected.

Bifurcation diagrams and comments For the spring-pendulum in 1 : 2 resonance, the iso-energetic Poincaré map is shown in Fig. 2.3 for various values of the detuning parameter. Its main bifurcation occurs at $a_1 \approx 0.69$, where an elliptic equilibrium disappears. To find analytic expressions for this bifurcation value, first the planar normal form is computed, which turns out to be

$$x(x^2 + y^2) + u_1 x + u_2 y^2.$$

Here u_i are the unfolding parameters. Its bifurcation diagram is depicted in Fig. 2.4. The curve $u_1 = 0$ corresponds to a saddle-center bifurcation, while $u_1 + 3u_2^2 = 0$ corresponds to a Hamiltonian pitchfork bifurcation. Using the reparametrizations, it turns out that the bifurcation curve $u_1 = 0$ does not correspond to bifurcations in the original system. The other curve does, however. Fig. 2.5 gives the bifurcation diagram in original parameters. Grey areas are portions of the parameter- or phase-plane that do not correspond to a configuration of the original system.

Table 2.2 Comparison of bifurcation values obtained numerically, and by means of the pull-back of the bifurcation curve, (2.14)

H	a_1	a_2	a_3	$\lambda_{\text{measured}}$	$\lambda_{\text{predicted}}$
0.01	0.5385	0.07	0.001	0.020	0.018
0.01	0.463	0.07	0.001	0.020	0.018
0.001	0.51255	0.07	0.001	0.00200	0.00198
0.001	0.4876	0.07	0.001	0.00200	0.00199
0.001	0.5365	0.2	0.001	0.00200	0.00201
0.001	0.465	0.2	0.001	0.00200	0.00198

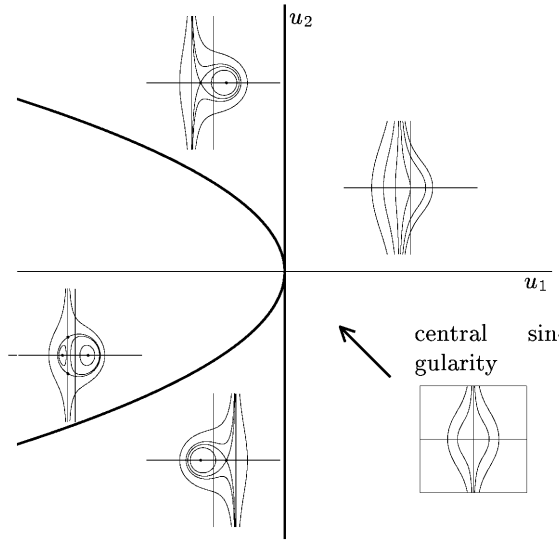

Fig. 2.4 Bifurcation diagram of the $\mathbb{Z}_2$-invariant hyperbolic umbilic $x(x^2+y^2)+u_1x+u_2y^2$. Across the bifurcation lines saddle-center bifurcations occur. Across the parabola $u_1+3u_2^2=0$ a Hamiltonian pitchfork bifurcation occurs due to $\mathbb{Z}_2$ symmetry.

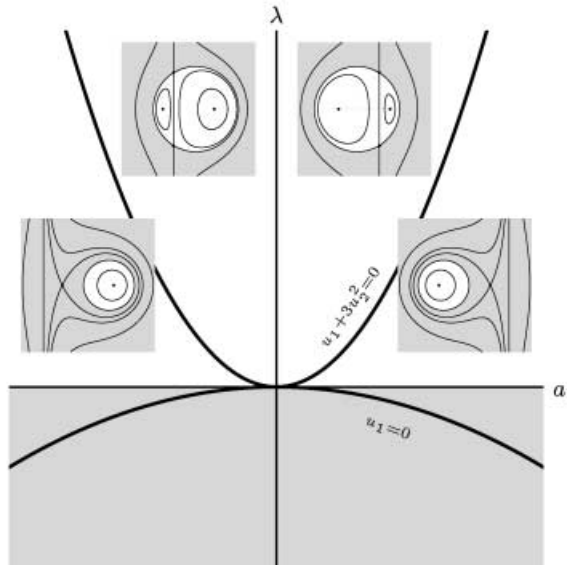

Fig. 2.5 Bifurcation diagram of the planar reduced system H^r, obtained from pulling back the bifurcation diagram of Fig. 2.4 to original coordinates. Grey areas denote portions of phase- or parameter-space that do not correspond to phase points of the original system.

3 Method II: The energy-momentum map

In this chapter we apply the energy–momentum map reduction method to the same class of systems as in Chap. 2, namely two degree-of-freedom systems with optional symmetry, near equilibrium and close to resonance. We calculate the tangent space and nondegeneracy conditions for the 1:2, 1:3 and 1:4 resonances starting from a Birkhoff normalized Hamiltonian. The case of the spring-pendulum close to 1:2 resonance is treated in more detail. We arrive at a polynomial model which is different from the one found in Chap. 2, and which has an additional saddle–node bifurcation.

3.1 Introduction

Several methods for analyzing Hamiltonian systems around resonance are available. One is the planar reduction method [BCKV95, BCKV93] of the previous chapter, but many more are available, see e.g. [Arn93a, GMSD95, SV85, Sch74] and references there. This chapter uses a method introduced in [Dui84, Sch74]. Just as the planar reduction method of Chap. 2, it uses the Birkhoff procedure followed by symmetric reduction. The singularity theory used subsequently is different, however, and uses left-right transformations to normalize a certain map from phase space to $\mathbb{R}^2$, the *energy–momentum map*. The remainder of the approach is again similar to that of Chap. 2: From a normal form of the map we compute bifurcation curves, and by explicitly computing the singularity theory transformations these are pulled back to original coordinates and parameters.

The algorithms we use to compute the reparametrizations (and the bifurcation curves) are closely related to those used in the previous chapter. Both Kas and Schlessinger's algorithm and the division algorithm have their counterparts in the present setting. With the planar reduction method of the previous chapter, the tangent space is an ideal, leading to a division algorithm that could be borrowed from Gröbner basis theory with little modification. The energy–momentum map method leads to a more complicated tangent space. These complications surface again in the division algorithm. In fact the complications were such that modifying the previous approach in an *ad-hoc* manner turned out to be infeasible. Instead we used a structured approach that brought, amongst others, Gröbner bases and standard bases for both tangent spaces into a common framework. Within this framework it was possible to derive the required division

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algorithm systematically. These results are described in Chaps. 6 and 7, and are applied in this chapter.

We now give an overview of the energy–momentum map method. For a general outline of this (and other) reduction methods, see the introduction to Chap. 2; here we shall be brief. After the Birkhoff procedure and truncation (or, modulo a flat perturbation), the system acquires an $\mathbb{S}^1$ symmetry, with associated conserved quantity H_2. We then construct the energy–momentum map $\boldsymbol{E}$, mapping phase space to $\mathbb{R}^2$. Its first component is the Hamiltonian, and the second component is the conserved quantity H_2. This map encodes information about the dynamics of Birkhoff-normalized system: Its fibers are invariant manifolds of the system, with singular points corresponding to periodic orbits. Both the fibers and singular points of $\boldsymbol{E}$ are smoothly deformed under the group of near-identity left-right transformations (B, A), with group operation $(B', A') * (B, A) = (B' \circ B, A \circ A')$, and action

$$\boldsymbol{E} \mapsto B \circ \boldsymbol{E} \circ A.$$

Since after the Birkhoff procedure the system has a (formal) circle symmetry, the map $\boldsymbol{E}$ has this symmetry too. It is necessary to do the singularity theory inside the space of symmetric mappings, since the orbit of $\boldsymbol{E}$ under left-right transformations has infinite codimension in the general space. Hence, we must also restrict to right-transformations A that commute with the symmetry. Instead of doing this explicitly, we reduce the symmetry by using circle-symmetric coordinates. For a 2 degree-of-freedom system, we thus can reduce from a circle-invariant map on $\mathbb{R}^4$, to a map on $\mathbb{R}^3$ that respects a certain algebraic relation between the variables.

Just as in the planar reduction method, dynamical conjugacy is lost with arbitrary (circle-equivariant but non-symplectic) right-transformations. However, if the system has 2 degrees of freedom, it lives on a 4-dimensional phase space, and nondegenerate fibers of $\boldsymbol{E}$ are 2-dimensional circle-invariant manifolds. After symmetry reduction we get 1-dimensional dynamically invariant manifolds, i.e., orbits of the reduced system. A universal deformation of $\boldsymbol{E}$ (i.e., a transversal section to its orbit under left-right transformations) can be related to a such a system that is *equivalent*, i.e., conjugate modulo a time-reparametrization, to the reduced Birkhoff-normalized system, again, just as in the planar reduction method.

In contrast to that method, it is not easy to de-reduce to the full system after normalization, because the fibers $H_2 =$ constant are not preserved by the normalizing transformations (even though, after [Dui84], the transformations we use preserve $\boldsymbol{E}$'s second component). The present method therefore seems less suited for studying the flat perturbations; see [BCKV95, BCKV93] for more remarks. Our current goal remains to pull back bifurcation curves. Although the larger class of allowed transformations necessitates an extra calculation to obtain the H_2-level at the bifurcation point, this turns out to be straightforward in this case.

A summary of this chapter was published as [Lun99b].

3.2 Description of the method

This section describes the general energy–momentum map method in detail. It follows [Dui84] to great extent. To avoid cluttering the formulas, the system's dependence on parameters and coefficients is suppressed in this section. From Sect. 3.4 onwards this dependence will be explicitly taken into account again.

3.2.1 Birkhoff normalization

The first step is Birkhoff normalization. Assume the Hamiltonian H has vanishing linear part. Then, after truncation (or modulo a flat perturbation) the Birkhoff-normalized Hamiltonian H^n is $\mathbb{S}^1$-symmetric, and assumes the form

$$H^n = H_2^0 + f_0(\rho_1, \rho_2, \psi, \chi), \tag{3.1}$$

where we expressed the normalized Hamiltonian in the invariants

$$\rho_1 = z_1\bar{z}_1, \quad \rho_2 = z_2\bar{z}_2, \quad \psi = \frac{1}{2}(z_1^p\bar{z}_2^{|q|} + \bar{z}_1^p z_2^{|q|}), \quad \chi = \frac{1}{2\mathrm{i}}(z_1^p\bar{z}_2^{|q|} - \bar{z}_1^p z_2^{|q|}),$$

and $H_2^0 = \mathrm{i}\rho_1 + \mathrm{i}\omega\rho_2$, with $\omega = \frac{p}{q}$, is an integral of motion. Note that H^n, and thus also f_0, depend on several parameters. In the case of the $1:2$ and $1:-2$ resonances, a second symplectic normalization removes the dependence on χ. This is a consequence of the unique normal form for Hamiltonians for these resonances; see [SvdM92].

Proposition 3.1. *Suppose $p:q = 1:\pm 2$ and the coefficients of ψ and χ in H^n do not both vanish. Then there exists a symplectic coordinate transformation that brings the Hamiltonian (3.1) in the form*

$$H^N = H_2^0 + f_1(\rho_1, \rho_2, \psi). \tag{3.2}$$

For a proof, see Sect. 4.3.2. In [Dui84] the dependence on χ is removed by a non-symplectic transformation. The present method leads to a shorter calculation, since we can calculate with Hamiltonians instead of vector fields. A preliminary calculation suggested that this approach cannot be used for higher resonances.

Remark 3.2. *(Comparison with Proposition 2.2)* Note that we arrived at (3.2) without assuming a time-reversal symmetry on H. This may be contrasted to the result of Proposition 2.2, where this assumption was needed. However, the current result only holds for the $1:2$ resonance, whereas Proposition 2.2 is more general.

3.2.2 Circle-equivariant vector fields

In this section we calculate generators for the module of vector fields equivariant under the H_2^0-circle action, under which the system is invariant after Birkhoff normalization. We pick the main line of the argument up again in Sect. 3.2.3.

From now on, we work in the ring of formal power series in the fundamental invariants, $R = \mathbb{R}[[\rho_1, \rho_2, \psi, \chi]]$, instead of $\mathbb{R}[[z_1, \bar{z}_1, z_2, \bar{z}_2]]$. There is one relation between the variables, namely

$$\psi^2 + \chi^2 - \rho_1^p \rho_2^q = 0. \tag{3.3}$$

For the $1 : \pm 2$ resonance cases, we can restrict to $\mathbb{R}[[\rho_1, \rho_2, \psi]]$ right away, and use the relation $\psi^2 - \rho_1^p \rho_2^q = 0$. We first deal with the general case.

Arbitrary resonances A circle-equivariant vector field in the variables $z_1, \bar{z}_1$, $z_2, \bar{z}_2$ corresponds to a vector field in $\rho_1, \rho_2, \psi, \chi$-space that leaves the relation (3.3) invariant. Such a vector field can always be written as

$$\boldsymbol{\alpha} = f_1 \frac{\partial}{\partial \rho_1} + f_2 \frac{\partial}{\partial \rho_2} + f_3 \frac{\partial}{\partial \psi} + f_4 \frac{\partial}{\partial \chi}.$$

The derivative of $\psi^2 + \chi^2 - \rho_1^p \rho_2^q$ in the direction of $\boldsymbol{\alpha}$ is

$$-p f_1 \rho_1^{p-1} \rho_2^q \; - \; q f_2 \rho_1^p \rho_2^{q-1} \; + \; 2 f_3 \psi \; + \; 2 f_4 \chi.$$

Requiring this to be an element of $\left\langle \psi^2 + \chi^2 - \rho_1^p \rho_2^q \right\rangle_R$ leads to conditions on the f_i and the exponents p, q, in turn leading to the following generators $\mathbf{v}_i$ of the R-module of equivariant vector fields:

	f_1	f_2	f_3	f_4
$\mathbf{v}_1$	$2\rho_1$		$p\psi$	$p\chi$
$\mathbf{v}_2$		$2\rho_2$	$q\psi$	$q\chi$
$\mathbf{v}_3$			χ	$-\psi$
$\mathbf{v}_4$	2ψ		$p\rho_1^{p-1}\rho_2^q$	
$\mathbf{v}_5$	2χ			$p\rho_1^{p-1}\rho_2^q$
$\mathbf{v}_6$		2ψ	$q\rho_1^p\rho_2^{q-1}$	
$\mathbf{v}_7$		2χ		$q\rho_1^p\rho_2^{q-1}$

These generators are independent over R, in the sense that there is no relation $\mathbf{v}_i = \sum_{j \neq i} f_j \mathbf{v}_j$ with $f_j \in R$, for any i. Below we shall need the derivative of H_2 along an arbitrary circle-invariant vector field $\boldsymbol{\alpha}$. So let $\boldsymbol{\alpha}$ be

$$\boldsymbol{\alpha} = v_1 \mathbf{v}_1 + \cdots + v_7 \mathbf{v}_7,$$

where $v_i \in R$, then the derivative of H_2 with respect to $\boldsymbol{\alpha}$ is

$$\boldsymbol{\alpha} H_2 \; = \; \boldsymbol{\alpha}(q\rho_1 + p\rho_2) \; = \; 2qv_1\rho_1 + 2pv_2\rho_2 + 2(qv_4 + pv_6)\psi + 2(qv_5 + pv_7)\chi. \tag{3.4}$$

Note that we wrote, for convenience, $H_2 = q\rho_1 + p\rho_2$, which differs from the quadratic part of H^N in (3.2) by a factor i/q; since we are interested in the location of critical points, this difference is immaterial.

The resonances 1 : ±2 An arbitrary vector field now takes the form $\boldsymbol{\alpha} = f_1\frac{\partial}{\partial\rho_1} + f_2\frac{\partial}{\partial\rho_2} + f_3\frac{\partial}{\partial\psi}$, and the condition that the derivative of the relation $\psi^2 - \rho_1^p\rho_2^q$ along $\boldsymbol{\alpha}$, which is $-pf_1\rho_1^{p-1}\rho_2^q - qf_2\rho_1^p\rho_2^{q-1} + 2f_3\psi$, is an element of the ideal generated by the relation, yields the following generators of the module of equivariant vector fields:

	f_1	f_2	f_3
$\mathbf{v}_1$	$2\rho_1$		$p\psi$
$\mathbf{v}_2$		$2\rho_2$	$q\psi$
$\mathbf{v}_4$	2ψ		$p\rho_1^{p-1}\rho_2^q$
$\mathbf{v}_6$		2ψ	$q\rho_1^p\rho_2^{q-1}$

The numbering of the generators is chosen to stress the relation with the previous case. An arbitrary circle-equivariant vector field, and the derivative of H_2 along it, now take the form

$$\begin{aligned} \boldsymbol{\alpha} &= v_1\mathbf{v}_1 + v_2\mathbf{v}_2 + v_4\mathbf{v}_4 + v_6\mathbf{v}_6, \\ \boldsymbol{\alpha}H_2 &= 2qv_1\rho_1 + 2pv_2\rho_2 + 2(qv_4 + pv_6)\psi. \end{aligned} \tag{3.5}$$

3.2.3 The energy–momentum map

We now introduce the main object of interest, the energy–momentum map:

$$\boldsymbol{E}: \quad \mathbb{R}^4 \to \mathbb{R}^2: \quad (\rho_1, \rho_2, \psi, \chi) \mapsto (H^n, H_2^0).$$

Here H^n is the Hamiltonian in Birkhoff normal form, depending on the parameters, and H_2^0 is the quadratic part of H^n when the parameters vanish (see also Chap. 4, remark 4.5). Note that the relation $\psi^2 + \chi^2 - \rho_1^p\rho_2^q = 0$ is supposed to hold, so that it suffices to define $\boldsymbol{E}$ on the variety of points satisfying the relation, but we do not make this explicit in the notation.

The fibers of $\boldsymbol{E}$ are invariant under the flow of H^n. They are smoothly deformed by left-right transformations (A, B), where $A : \mathbb{R}^4 \to \mathbb{R}^4$ are circle-equivariant near-identity diffeomorphisms on $\mathbb{R}^4$, and $B : \mathbb{R}^2 \to \mathbb{R}^2$ are arbitrary planar near-identity diffeomorphisms, with action

$$(A, B): \quad \boldsymbol{E} \mapsto B \circ \boldsymbol{E} \circ A.$$

In the context of unfoldings, the left-right transformation (A, B) depends on parameters, and is the identity mapping at the origin in parameter space. The set of invertible left-right transformations forms a group, with the obvious composition as group operation. To find a universal deformation of $\boldsymbol{E}$, we look at the tangent space, at $\boldsymbol{E}$, to the orbit of $\boldsymbol{E}$ under the action of the group. This tangent space is

$$T_{\boldsymbol{E}} = \{(\boldsymbol{\alpha}H + \beta_1(H, H_2), \boldsymbol{\alpha}H_2 + \beta_2(H, H_2))\},$$

where $\boldsymbol{\alpha}$ runs over the circle-equivariant vector fields, and β_i are arbitrary functions of 2 variables, see Sect. 5.4.

The set $T_{\boldsymbol{E}}$ is a subset of $R \oplus R$, the space of maps from $\mathbb{R}^n$ to $\mathbb{R}^2$ (recall that $R = \mathbb{R}[[\rho_1, \rho_2, \psi, \chi]]$). It is possible to reduce $T_{\boldsymbol{E}}$ to a subset of R in the following way. Consider the map T going from the tangent space of the group of left-right transformations at the identity element, to $R \oplus R$, defined by

$$T: \quad (\boldsymbol{\alpha}, \beta_1, \beta_2) \mapsto (\boldsymbol{\alpha}H + \beta_1(H, H_2), \boldsymbol{\alpha}H_2 + \beta_2(H, H_2)).$$

We have $\operatorname{Im} T = T_{\boldsymbol{E}}$. From (3.4) it follows that the image of the second component of T, denoted by T_2, maps surjectively into R. Therefore, since T is a linear map, the codimension of $T_{\boldsymbol{E}}$ in $R \oplus R$ is the codimension in R of the image of T_1, restricted to the kernel of T_2. In other words, to compute the codimension of $T_{\boldsymbol{E}}$, we may restrict to left-right transformations that do not alter the second component of $\boldsymbol{E}$, and look at how those transformations change its first component. If the codimension of $T_1(\ker T_2) \subseteq R$ is finite, and $t_1, \ldots, t_d$ are complementing elements, then the elements $(t_1, 0), \ldots, (t_d, 0)$ span a complement of $T_{\boldsymbol{E}}$. By the results of Chap. 5, this immediately gives a universal deformation of $\boldsymbol{E}$, say F, namely $F : \mathbb{R}^4 \oplus \mathbb{R}^d \to \mathbb{R}^2 : (x, u) \mapsto \boldsymbol{E} + u_1 \cdot (t_1, 0) + \cdots + u_d \cdot (t_d, 0)$ We shall denote the space $T_1(\ker T_2)$ by $T^r_{\boldsymbol{E}}$, the *reduced* tangent space to $\boldsymbol{E}$. This reduced tangent space takes the following form:

$$T^r_{\boldsymbol{E}} := \{\boldsymbol{\alpha}H + \beta_1(H, H_2) | \boldsymbol{\alpha}H_2 + \beta_2(H, H_2) = 0\}, \tag{3.6}$$

where $\boldsymbol{\alpha}$ runs over the circle invariant vector fields, and β_i over the arbitrary functions in two variables. From (3.4) it follows that $\boldsymbol{\alpha}H_2$ vanishes at the origin, hence β_2 can be written as $\beta_2(x, y) = x\gamma_1(x, y) + y\gamma_2(x, y)$. Also from (3.4) it follows that for every function f vanishing at the origin, there exists a vector fields $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}H_2 = f$. In particular, let $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ be defined by

$$\begin{aligned} \boldsymbol{\alpha}_1 H_2 &= H, \\ \boldsymbol{\alpha}_2 H_2 &= H_2. \end{aligned}$$

Then, writing $\beta_2(H, H_2) = H\gamma_1(H, H_2) + H_2\gamma_2(H, H_2)$, the reduced tangent space can be written as

$$T^r_{\boldsymbol{E}} = \{\boldsymbol{\alpha}H + \beta_1(H, H_2) | \left(\boldsymbol{\alpha} + \gamma_1(H, H_2)\boldsymbol{\alpha}_1 + \gamma_2(H, H_2)\boldsymbol{\alpha}_2\right) H_2 = 0\},$$

and with a change of variables $\boldsymbol{\alpha}' = \boldsymbol{\alpha} + \gamma_1\boldsymbol{\alpha}_1 + \gamma_2\boldsymbol{\alpha}_2$ it becomes

$$T^r_{\boldsymbol{E}} = \{\boldsymbol{\alpha}'H + f_1\gamma_1(H, H_2) + f_2\gamma_2(H, H_2) + \beta_1(H, H_2) | \boldsymbol{\alpha}'H_2 = 0\},$$

where we wrote

$$f_i = -\boldsymbol{\alpha}_i H.$$

Writing this differently, we arrive at the following:

Proposition 3.3. *The codimension of $T_{\boldsymbol{E}}$ is equal to the codimension of $T^r_{\boldsymbol{E}} := T_1(\ker T_2) \subseteq R$. This set can be written in the form*

$$T^r_{\boldsymbol{E}} = J + \{1, f_1, f_2\}\mathbb{R}[[H, H_2]], \tag{3.7}$$

where f_i are as defined above, and J is the ideal

$$J = \{\boldsymbol{\alpha}H | \boldsymbol{\alpha}H_2 = 0\}.$$

Generators of J From (3.4) it follows that the set of equivariant vector fields $\boldsymbol{\alpha}$ that leave H_2 fixed is generated, as an R-module, by

$$\begin{gathered}\mathbf{v}_3, \quad p\rho_2\mathbf{v}_1 - q\rho_1\mathbf{v}_2, \quad p\mathbf{v}_4 - q\mathbf{v}_6, \quad p\mathbf{v}_5 - q\mathbf{v}_7, \\ \psi\mathbf{v}_1 - \rho_1\mathbf{v}_4, \quad \chi\mathbf{v}_1 - \rho_1\mathbf{v}_5, \quad \psi\mathbf{v}_2 - \rho_2\mathbf{v}_6, \quad \chi\mathbf{v}_2 - \rho_2\mathbf{v}_7, \quad \chi\mathbf{v}_4 - \psi\mathbf{v}_6.\end{gathered}$$

The last five generators are actually multiples of $\mathbf{v}_3$, taking the relation into account. The first four, acting on H, yield the following generators of J:

$$\begin{aligned}h_0 &:= \psi^2 + \chi^2 - \rho_1^p\rho_2^q, \\ h_1 &:= \chi\frac{\partial H}{\partial \psi} - \psi\frac{\partial H}{\partial \chi}, \\ h_2 &:= 2\rho_1\rho_2\left(p\frac{\partial H}{\partial \rho_1} - q\frac{\partial H}{\partial \rho_2}\right) + \left(p^2\rho_2 - q^2\rho_1\right)\left(\chi\frac{\partial H}{\partial \chi} + \psi\frac{\partial H}{\partial \psi}\right), \\ h_3 &:= 2\psi\left(p\frac{\partial H}{\partial \rho_1} - q\frac{\partial H}{\partial \rho_2}\right) + \left(p^2\rho_1^{p-1}\rho_2^q - q^2\rho_1^p\rho_2^{q-1}\right)\frac{\partial H}{\partial \psi}, \\ h_4 &:= 2\chi\left(p\frac{\partial H}{\partial \rho_1} - q\frac{\partial H}{\partial \rho_2}\right) + \left(p^2\rho_1^{p-1}\rho_2^q - q^2\rho_1^p\rho_2^{q-1}\right)\frac{\partial H}{\partial \chi}.\end{aligned} \tag{3.8}$$

Note that we added the relation h_0, so that we can work in the free ring $R = \mathbb{R}[[\rho_1, \rho_2, \psi, \chi]]$.

Generators of J in the $1 : \pm 2$-resonance case Sect. 3.2.3 is valid for the $1 : \pm 2$ resonance case, up to the computation of generators of J. We give the details for that case here. Referring to (3.5), we find the following generators of the R-module of vector fields that leave H_2 invariant:

$$p\rho_2 V_1 - q\rho_1 V_2, \quad pV_4 - qV_6, \quad \psi V_1 - \rho_1 V_4$$

(where $p = 1, q = \pm 2$). The last generator is zero modulo the relation $\psi^2 - \rho_1^p\rho_2^q = 0$. The other two generators, acting on H, together with the relation, yield the following generators of J:

$$\begin{aligned}h_0 &:= \psi^2 - \rho_1^p\rho_2^q, \\ h_1 &:= 2\rho_1\rho_2\left(p\frac{\partial H}{\partial \rho_1} - q\frac{\partial H}{\partial \rho_2}\right) + (p^2\rho_2 - q^2\rho_1)\psi\frac{\partial H}{\partial \psi}, \\ h_2 &:= 2\psi\left(p\frac{\partial H}{\partial \rho_1} - q\frac{\partial H}{\partial \rho_2}\right) + (p^2\rho_1^{p-1}\rho_2^q - q^2\rho_1^p\rho_2^{q-1})\frac{\partial H}{\partial \psi}.\end{aligned} \tag{3.9}$$

3.2.4 Removing the χ-dependence

At this point we introduce a nondegeneracy condition, namely that $\frac{\partial H}{\partial \phi}$ and $\frac{\partial H}{\partial \psi}$ do not both vanish at the origin. (We would have needed this condition for other reasons later on anyway.) Under this condition it is possible to reduce the

dimension of the phase space by 1. This simplifies the computations. Again, see [Dui84].
Note that the module generated by V_4, $p\rho_2 V_2 - q\rho_1 V_3$, $pV_5 - qV_7$ and $pV_6 - qV_8$ is invariant under rotations in the ψ, χ-plane. This is obvious from the definition of the module, since H_2 is invariant under such rotations – indeed, H_2 is independent of ψ and χ.

By a rotation in the ψ, χ plane, we can arrange that $\frac{\partial H}{\partial \chi}(0) = 0$. Now assume that in this situation, $\frac{\partial H}{\partial \psi}(0) = d \neq 0$. By a further coordinate transformation, H can be made not to depend on χ at all, in the following way. Let

$$H = h_0 + \sum_{1<k<n} \tilde{f}_k(\psi) + \sum_{k \geq n} f_k(\psi, \chi),$$

where $n \geq 2$, $h_0 \in \mathbb{R}[[\rho_1, \rho_2, \psi]]$, $\tilde{f}_k$ is homogeneous of degree k in ψ, and f_k is homogeneous of total degree k in ψ and χ, both with coefficients in $\mathbb{R}[[\rho_1, \rho_2]]$. Write

$$f_n = \tilde{f}_n + \chi g_n,$$

with $\tilde{f}_n$ not depending on χ, and let $\boldsymbol{\alpha}$ be the vector field

$$\boldsymbol{\alpha} = -\frac{g_n}{d}\left(\chi \frac{\partial}{\partial \psi} - \psi \frac{\partial}{\partial \chi}\right).$$

It has time-1 flow

$$\boldsymbol{\alpha}^1(\psi, \chi) = \left(\psi - \frac{g_n}{d}\chi + O(|\psi, \chi|^{2n}), \chi + \frac{g_n}{d}\psi + O(|\psi, \chi|^{2n})\right).$$

Since $\frac{\partial h_0}{\partial \psi}(0) = d$ and $\frac{\partial h_0}{\partial \chi}(0) = 0$, we have $h_0 \circ \boldsymbol{\alpha}^1 = h_0 - \chi g_n + O(|\psi, \chi|^{n+1})$, and therefore

$$H \circ \boldsymbol{\alpha}^1 = h_0 + \sum_{1<k<n} \tilde{f}_k(\psi) + \tilde{f}_n(\psi) + \sum_{k \geq n+1} f_k(\psi, \chi).$$

Continuing in this way, all dependence in H of χ is removed, without changing the form of (3.8).

From now on we write R for the ring $\mathbb{R}[[\rho_1, \rho_2, \psi]]$, and we add the relation $\psi^2 - \rho_1^p \rho_2^q$ to the generators of the ideal J so that J is generated by

$$\begin{aligned}
h_0 &= \psi^2 - \rho_1^p \rho_2^q, \\
h_1 &= 2\rho_1\rho_2 \left(p\frac{\partial H}{\partial \rho_1} - q\frac{\partial H}{\partial \rho_2}\right) + \left(p^2\rho_2 - q^2\rho_1\right)\psi\frac{\partial H}{\partial \psi}, \\
h_2 &= 2\psi\left(p\frac{\partial H}{\partial \rho_1} - q\frac{\partial H}{\partial \rho_2}\right) + \left(p^2\rho_1^{p-1}\rho_2^q - q^2\rho_1^p\rho_2^{q-1}\right)\frac{\partial H}{\partial \psi}.
\end{aligned}$$

Note that these expressions are exactly equal to (3.9) of the $1 : \pm 2$-resonance case, where the removal of χ was done differently (namely, by a symplectic transformation).

3.3 Application to several resonances

In this section we compute standard bases for the left-right tangent space of Hamiltonians around several resonances. As a result, we get independent confirmation of the calculations in [Dui84], and explicit non-degeneracy conditions.

It is assumed that the Hamiltonian is in Birkhoff normal form, so that it can be written as a function of the fundamental circle invariants, which for the $p : q$ resonance are

$$\rho_1 = z_1\bar{z}_1, \quad \rho_2 = z_2\bar{z}_2, \quad \psi = \frac{1}{2}(z_1^p\bar{z}_2^{|q|} + \bar{z}_1^p z_2^{|q|}), \quad \chi = \frac{1}{2\mathrm{i}}(z_1^p\bar{z}_2^{|q|} - \bar{z}_1^p z_2^{|q|}).$$

We further assume that the dependence on χ is removed. Using the relation $\psi^2 - \rho_1^p\rho_2^q = 0$ the variable ψ can be made to appear to first order at most. The coefficients associated to the lowest-order terms are given names as follows:

(3.10)
$$\begin{aligned} H = {} & d_1\rho_1 + d_2\rho_2 + d_3\psi + d_4\rho_1^2 + d_5\rho_1\rho_2 + d_6\psi\rho_1 + d_7\rho_2^2 + d_8\psi\rho_2 + d_9\psi^2 + \\ & d_{10}\rho_1^3 + d_{11}\rho_1^2\rho_2 + d_{12}\psi\rho_1^2 + d_{13}\rho_1\rho_2^2 + d_{14}\psi\rho_1\rho_2 + d_{15}\psi^2\rho_1 + d_{16}\rho_2^3 + \\ & d_{17}\psi\rho_2^2 + d_{18}\psi^2\rho_2 + d_{19}\psi^3 + d_{20}\rho_1^4 + d_{21}\rho_1^3\rho_2 + d_{22}\psi\rho_1^3 + d_{23}\rho_1^2\rho_2^2 + \\ & d_{24}\psi\rho_1^2\rho_2 + d_{25}\rho_1 v 2^3 + d_{26}\psi\rho_1\rho_2^2 + d_{27}\rho_2^4 + d_{28}\psi\rho_2^3. \end{aligned}$$

The reduced tangent space is of the form

$$T^r_{\boldsymbol{E}} = \langle h_0, \ldots, h_k\rangle + \{f_0, \ldots, f_l\}\mathbb{R}[[g_1, \ldots, g_m]].$$

Once the basis $(\{h_i\}, \{f_i\}, \{f_i\})$ is extended to a standard basis, the codimension and deformation directions can be deduced from the leading monomials of the generators: The deformation directions are the monomials *not* in the basis, and the codimension is their number. The coefficients of the generator's leading monomials should be nonzero, as otherwise the codimension increases. This leads to the nondegeneracy conditions.

It often happens that a certain leading monomial can be reached through several reduction paths. The associated coefficient may be different for each path. Therefore we tried all different paths, to make sure no spurious nondegeneracy conditions were included.

Remark 3.4. *(Rigorous codimensions)* This is a computational study, and the codimensions obtained below are not rigorous, since we truncate at a certain degree. They are a rigorous *lower* bound, however. All this is rather academic, since the codimensions below are established, by different means, in [Dui84]. A related remark is that the set of nondegeneracy conditions obtained may not be complete. Since a full (non-truncated) standard basis will contain infinitely many elements, as is easily seen, there may in principle be infinitely many (polynomial) nondegeneracy conditions. Indeed, in some cases we find more of such conditions than appear in [Dui84].

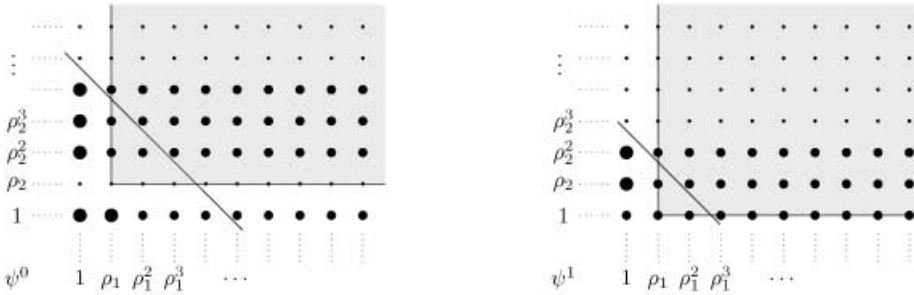


Fig. 3.1 Standard basis of LR-tangent space for the 1 : 2 resonance.

3.3.1 The 1:2 resonance

We use the expressions (3.9) for the generators of J, truncated at order 9 in the z_i variables. We use a grading of the monomials with $\deg(\rho_1) = \deg(\rho_2) = 2$, $\deg(\psi) = 3$, and the ordering satisfies $\psi^2 < \rho_1^3 < \rho_2^3$. The computation of the standard basis then yields the following leading monomials:

$$\begin{aligned}
\mathsf{LM}(\{h_i\}) &= \{\psi\rho_1, \rho_1\rho_2, \psi^2\},\\
\mathsf{LM}(\{g_i\}) &= \{\rho_1, \psi\},\\
\mathsf{LM}(\{f_i\}) &= \{1, \rho_1, \rho_2^2, \rho_2^3, \rho_2^4, \psi\rho_2, \psi\rho_2^2\}.
\end{aligned}$$

This standard basis is depicted graphically in Fig. 3.1. Monomials are represented by lattice points in $\mathbb{N}^3$. Since one of the ideal generators has leading monomial ψ^2, the interesting things happen in the ψ^0 and ψ^1 slices. The monomials in the ideal are those in the grey rectangles. Fat dots denote f_i-generators, and the medium-sized dots correspond to monomials obtained by multiplying an f_i and an element of the algebra. The diagonal line, finally, shows where we truncated. From the figure, it is seen that the codimension is 1, and that the deformation term that makes the Energy–Momentum map versal, is ρ_2. This is called the *detuning* term.

The requirement that, for each basis element, the coefficients of the leading monomials do not vanish, leads to the following nondegeneracy conditions:

$$\begin{aligned}
d_3 &\neq 0,\\
8d_3^2 - 2d_4 + d_5 + 4d_7 &\neq 0.
\end{aligned} \tag{3.11}$$

A *normal form* for the Hamiltonian behind this tangent space is a simple function (a polynomial, for example) with few parameters, but which nevertheless captures all bifurcations. A sufficient condition for this is to include all terms whose parameters appear in the nondegeneracy conditions; perturbations in the other terms will not change the type of the singularity. This condition is generally not necessary. However, for our purposes it is useful to be able to choose

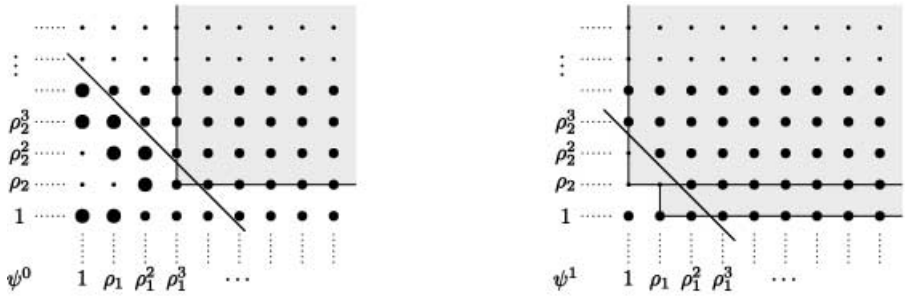


Fig. 3.2 Standard basis of LR-tangent space for the 1 : 3 resonance.

parameters such that the normalizing transformation is the identity in leading order. It turns out that e.g.

$$H = \frac{1}{2}\rho_1 + \left(\frac{1}{4} + \mu_1\right)\rho_2 + d_3\psi + d_5\rho_1\rho_2 \tag{3.12}$$

is a suitable normal form. Here d_3 and d_5 are (fixed) parameters, and μ_1 is the *detuning parameter*.

3.3.2 The 1:3 resonance

For this resonance, the reduced tangent space was calculated up to order 8 in the z_i variables. The variable ψ has degree 4, so that it appears at most to second degree. We use a monomial ordering satisfying $\psi < \rho_2^2 < \rho_1^2$. With this ordering the leading monomials of the standard basis are

$$\begin{aligned}
\mathsf{LM}(\{h_i\}) &= \{\psi\rho_1, \psi^2, \psi\rho_2, \rho_1^3\rho_2\},\\
\mathsf{LM}(\{g_i\}) &= \{\rho_1, \psi\},\\
\mathsf{LM}(\{f_i\}) &= \{1, \rho_1, \rho_1^2\rho_2, \rho_1^2\rho_2^2, \rho_1\rho_2^3, \rho_1\rho_2^2, \rho_2^4, \rho_2^3\}.
\end{aligned}$$

A graphical representation of this standard basis is given in figure 3.2. Now the codimension is 3, with corresponding deformation directions ρ_2, $\rho_1\rho_2$ and ρ_2^2. The nondegeneracy conditions for this resonance are somewhat complicated, so we introduce the following abbreviations:

$$\begin{array}{lll}
a = \frac{1}{2}(d_{10} + d_{11}) & c = d_4 - d_5 - 3d_7 & e = \frac{1}{2}(d_6 + d_8)\\
b = \frac{1}{2}(d_{10} - d_{11}) & d = 2d_4 - 3d_5 & f = -2d_6 - 3d_8
\end{array}$$

In terms of these quantities, the nondegeneracy conditions are the following:

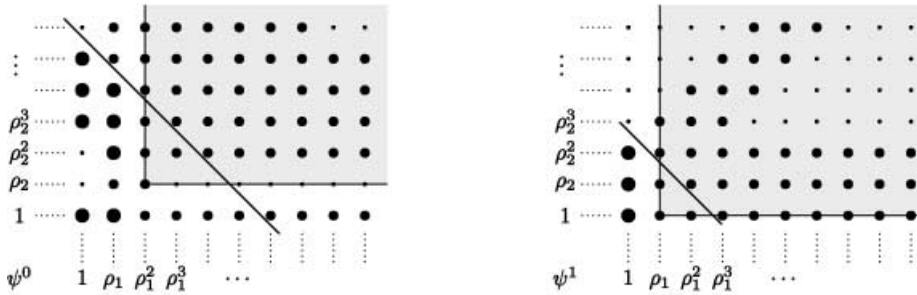


Fig. 3.3 Standard basis of LR-tangent space for the 1 : 4 resonance.

$d_3 \neq 0$,
$d_4 + 3d_5 - 27d_7 \neq 0$,
$c \neq 0$,
$d \neq 0$,
$3^6(9c+2d)d_3^4 + 216d(12c^2+3cd+2d^2)d_3^2 + 16d^2(9c^3-6c^2d-3cd^2+2d^3) \neq 0$,
$216bd_3^3 - 54ded_3^2 + 4(6bc^2+acd-19bcd+8bd^2-3cdd_{13})d_3 +$
$8(d-c)d(6ce-de+cf) \neq 0$,
$5832bd_3^5 - 1458ded_3^4 + 27(168bc^2+10acd-228bcd+64bd^2-21cdd_{13})d_3^3$
$-108d(12c^2e-25cde+4d^2e-c^2f-2cdf)d_3^2$
$+4(216bc^4+36ac^3d-612bc^3d-38ac^2d^2+580bc^2d^2+10acd^3-228bcd^3+$
$+32bd^4-108c^3dd_{13}+99c^2d^2d_{13}-21cd^3d_{13})d_3$
$-16(3c-2d)(2c-d)(3c-d)d(6ce-de+cf) \neq 0$.

A normal form for this singularity is for example

$$H = \frac{1}{2}\rho_1 + \left(\frac{1}{6}+\mu_1\right)\rho_2 + d_3\psi + d_4\rho_1^2 + (d_5+\mu_2)\rho_1\rho_2 + (d_7+\mu_3)\rho_2^2 + d_{10}\rho_1^3 + d_{13}\rho_1\rho_2^2.$$

3.3.3 The 1:4 resonance

For this resonance we again chose the obvious grading, $\deg(\rho_1) = \deg(\rho_2) = 2$ and $\deg(\psi) = 5$, and the monomial order defined by $\psi^2 < \rho_2^5 < \rho_2^5$. The truncation degree was set at 11, resulting in the following standard basis:

$$\begin{aligned}
\mathsf{LM}(\{h_i\}) &= \{\rho_1^2\rho_2, \psi\rho_1, \psi^2\},\\
\mathsf{LM}(\{g_i\}) &= \{\rho_1, \rho_1\rho_2\},\\
\mathsf{LM}(\{f_i\}) &= \{1, \rho_1, \rho_1\rho_2^2, \rho_1\rho_2^3, \rho_1\rho_2^4, \rho_2^5\rho_2^4, \psi\rho_2^2, \psi\rho_2, \rho_2^3, \psi\}.
\end{aligned}$$

The codimension is 2, with deformation terms ρ_2 and ρ_2^2. The nondegeneracy conditions are the following:

$$
\begin{aligned}
&d_3 \neq 0, .\\
&d_4 - 2d_5 \neq 0,\\
&3d_4 - 8d_5 - 16d_7 \neq 0,\\
&d_4 - 16d_7 \neq 0,\\
&d_5 - 8d_7 \neq 0,\\
&d_4 - 4d_5 + 16d_7 \neq 0,\\
&5d_4^2 + 32d_5^2 - 128d_5d_7 + 256d_7^2 + 32d_4d_7 - 24d_4d_5 \neq 0.
\end{aligned}
$$

A normal form is for example

$$H = \frac{1}{2}\rho_1 + \left(\frac{1}{8} + \mu_1\right)\rho_2 + d_3\psi + d_4\rho_1^2 + d_5\rho_1\rho_2 + (d_7 + \mu_2)\rho_2^2.$$

3.4 Spring-pendulum in 1:2 resonance

In this section we shall analyze the spring-pendulum system in the 1 : 2 resonance, by exhibiting a polynomial model of the Energy–Momentum map, and computing the reparametrizations that connect the model to the original system. The bifurcation analysis of the model leads to a bifurcation diagram, together with a 'catalog' of possible dynamics as depicted in phase diagrams. Using the explicit reparametrization the bifurcation curves are pulled back to the space of the spring pendulum's original parameters, giving full information as to which subset of the full set of possible dynamics actually occurs, and where bifurcations take place in terms of the original parameters.

This section's analogue for the planar reduction case is section 2.3, but this section is organized slightly differently. We start with a bifurcation analysis of the polynomial normal form, and we give pictures of the bifurcation and phase space diagrams. This is followed by the computation of the explicit reparametrizations, of which we only give the results. These are then used to pull back the bifurcation curves to the original parameter space.

3.4.1 Bifurcation analysis of the 1:2-resonant normal form

In Sect. 2.3 we encountered a third-order polynomial normal form, with a rather simple bifurcation structure. The normal form for this case is less simple, involving degree-four terms, which complicates the computations. The normal form for the 1 : 2-resonant system is:

$$\boldsymbol{E}^{\mu} = (H^{\mu}, H_2) = (H_2 + \mu\rho_2 + a\psi + b\rho_1\rho_2, H_2)\,, \tag{3.13}$$

see Sect. 3.3.1. Here we wrote H_2 for the quadratic part $\frac{1}{2}\rho_1 + \frac{1}{4}\rho_2$ as usual; a and b are fixed coefficients, and μ is the deformation parameter; see (3.10), (3.12). In Hamiltonian polar coordinates $z_i = r_i e^{2\pi i\phi_1}$, the fundamental invariants become $\rho_i = r_i^2$ and $\psi = r_1 r_2^2 \cos(\phi_1 - 2\phi_2)$. These variables are used in the sequel.

We restrict to the flow-invariant level sets $2H_2 = \lambda$, for small λ, which corresponds to small-energy deviations from the elliptic equilibrium. The function H^μ restricted to such level sets has critical points precisely where $\boldsymbol{E}^\mu$ has critical points, namely where $\mathrm{grad}(H_2)$ is parallel to $\mathrm{grad}(H^\mu)$. A short calculation shows that a necessary condition[1] for this is $\sin(\phi_1 - 2\phi_2) = 0$. It is now convenient to introduce $\epsilon := \cos(\phi_1 - 2\phi_2)$, so that $\epsilon = \pm 1$. Now $\mathrm{grad}(H_2)$ and $\mathrm{grad}(H^\mu)$ are vectors in $\mathbb{R}^2 \ni (r_1, r_2)$, and such vectors are parallel if their outer product vanishes. The resulting curve of critical points is

$$r_2 \left(-2\mu r_1 - 2a\epsilon r_1^2 - 2b r_1^3 + \left(\tfrac{1}{2}a\epsilon + b r_1\right) r_2^2\right) = 0. \tag{3.14}$$

This implies that $r_2 = 0$ is always a critical point; it corresponds to the 'short periodic orbit' that always exists.

Equation (3.14), with the solution $r_2 = 0$ divided out, can be solved for r_2^2. Obviously, critical points (dis)appear when the solution r_2^2 of (3.14) passes through 0. This leads to a bifurcation curve in the μ, λ-plane implicitly given by

$$(\mu + 2b\lambda)^2 - 2a^2\lambda = 0. \qquad \textit{(pitchfork bifurcation)} \tag{3.15}$$

The parabolic equation (3.15) has solutions for nonnegative λ only. Since the critical points emanate from $r_2 = 0$ which itself is a critical point throughout the bifurcation, this is a pitchfork bifurcation. The bifurcation occurs at the point

$$(r_1, r_2, \psi, \chi) = \left(\left|\frac{1}{a}\mu + \frac{b}{a^3}\mu^2 + \frac{2b^2}{a^5}\mu^3\right|, 0, 0, 0\right) + O(\mu^4). \tag{3.16}$$

Other bifurcations occur when the curve of critical points (3.14) in the r_1, r_2-plane become tangent to level curves of H_2. When this happens, a small change in the level λ will (generically) create or destroy two critical points. One of these will be a saddle, the other a center or a *node*, so that this is a saddle–node bifurcation. The tangency condition is that $\mathrm{grad}(H_2)$ and the gradient of the left-hand-side of (3.14) be parallel, and using the outer product once more we get the bifurcation equation

$$r_2 \left(2\mu + 6a\epsilon r_1 + 10 b r_1^2 - b r_2^2\right) = 0. \tag{3.17}$$

Bifurcations occur at simultaneous solutions, in r_1, r_2, μ, λ, of $2H_2 = \lambda$, (3.14) and (3.17). This solution curve can be found by computing a Gröbner basis with respect to an elimination term order (see Sect. 6.2.1) of the ideal generated by these three polynomials. Since only three generators are involved, we can get the same result (with, essentially, the same computation) using resultants.[2] Computing the resultant of the pair $2H_2 - \lambda$, (3.14) and of (3.14), (3.17) with respect

[1] We exclude the degenerate case $r_1 = r_2 = 0$.

[2] The resultant of two polynomials, with respect to the variable x, is their GCD with respect to the divisibility condition $x^a | x^b \Leftrightarrow a \leq b$, i.e., it vanishes iff the polynomials have a common root in x.

to r_2, eliminates this variable, and reduces the system of 3 polynomial equations to a system involving only 2. The resultant with respect to r_1 subsequently reduces this system to the following cubic curve in the μ, λ-plane, along which the saddle–node bifurcations occur:

(3.18)
$$\begin{aligned} &27a^4\lambda + 9a^2b^2\lambda^2 + 32b^4\lambda^3 - 126a^2b\lambda\mu \\ &-96b^3\lambda^2\mu + 9a^2\mu^2 + 96b^2\lambda\mu^2 - 32b\mu^3 = 0, \end{aligned} \qquad \textit{(saddle–node bifurcation)}$$

where we used that $\epsilon^2 = 1$, and we first divided out two factors r_2 corresponding to the pitchfork bifurcation. Note that the cubic curve is independent of ϵ now. Also note that the curve has quadratic contact with the line $\lambda = 0$ in $\mu = 0$. By expanding around 0 it turns out that the curve has no solutions $\lambda > 0$ around $(\lambda, \mu) = (0, 0)$, for all values of the coefficients a, b (provided $b \neq 0$). It has a critical point at $(\mu, \lambda) = (\frac{5a^2}{16b}, \frac{-a^2}{16b^2})$.

Not all points on the cubic correspond to saddle–node bifurcations. The reason is that due to the constraint $2H_2 = \lambda = \rho_1 + \frac{1}{2}\rho_2 = r_1^2 + \frac{1}{2}r_2^2$, the solutions for the r_i may be imaginary. The 'turning points' on the cubic, where real solutions turn into imaginary ones, is where either r_1^2 or r_2^2 pass through 0. From (3.14) it follows that $r_1 \neq 0$ at critical points (provided $\lambda \neq 0$). So we turn our attention to the equation $r_2 = 0$. The curve of pitchfork bifurcations (3.15) is just the curve where critical points satisfy $r_2 = 0$, hence the turning points are intersections of the cubic (3.18) with the parabola (3.15).

To compute these intersection points we compute the resultant of the two curves. With respect to λ this is

$$4096b^{10}\mu^2(-a^2 + 4b\mu)^2(153a^4 - 352a^2b\mu + 256b^2\mu^2).$$

From this we conclude that (3.18) and (3.15) have quadratic contact at $(\mu, \lambda) = (0, 0)$ and $(\mu, \lambda) = (\frac{a^2}{4b}, \frac{a^2}{4b^2})$. The last factor is quadratic in μ, with discriminant $-32768a^4b^2$. Provided that neither a nor b vanishes, this does not contribute additional (real) zeros.

Since the part of the cubic connecting the turning points $(\mu, \lambda) = (0, 0)$ and $(\frac{a^2}{4b}, \frac{a^2}{4b^2})$ passes through the $\lambda < 0$ region, namely through the critical point $(\frac{5a^2}{16b}, \frac{-a^2}{16b^2})$, we conclude that saddle–node bifurcations occur only for

$$\lambda > \frac{a^2}{4b^2}. \qquad \textit{(condition for saddle–node bifurcation)}$$

In order to get a useful expression for μ in terms of λ, we expand the cubic (3.18) around the turning point. Writing $\Lambda = \lambda - \frac{a^2}{4b^2}$ we get

$$\mu = \frac{a^2}{4b} + \frac{b^3}{3a^2}\Lambda^2 - \frac{10b^5}{27a^4}\Lambda^3 + \frac{47b^7}{81a^6}\Lambda^4 + O(\Lambda^5). \qquad \textit{(saddle–node bifurcation)}$$

We also computed the bifurcation locus, around the turning point. Writing $\mathrm{M} := \mu - \frac{a^2}{4b}$, then saddle–node bifurcations occur for $\mathrm{M}/b \geq 0$ at points which have the following expansion in $\sqrt{\mathrm{M}}$:

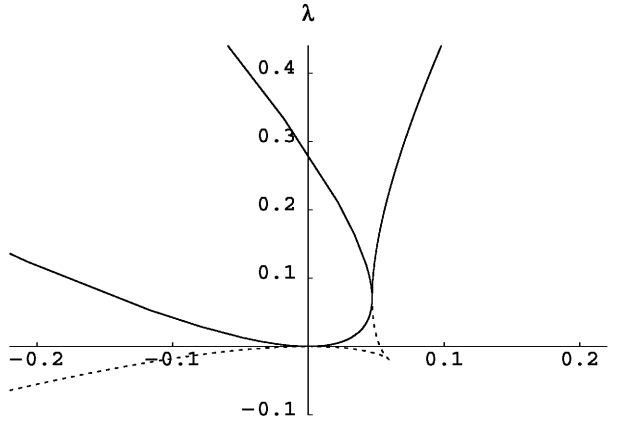


Fig. 3.4 Bifurcation curves of the normal form (3.13), for $a = \frac{1}{4}$ and $b = \frac{1}{3}$. The parabola is a curve of pitchfork bifurcations; along the singular cubic curve saddle–node bifurcations occur. Dashed segments correspond to non-physical states of the system (λ out of bounds, leading to imaginary values of state variables).

(3.19)

$$
\begin{aligned}
(r_1, r_2^2, \psi, \chi) = \Bigg(& \operatorname{sign}(b) \left(\tfrac{|a|}{2b} + \sqrt{\tfrac{\mathrm{M}}{3b}} + \tfrac{4\mathrm{M}}{9|a|} + \tfrac{40\sqrt{b\mathrm{M}^3}}{27\sqrt{3}a^2} + \tfrac{512b\mathrm{M}^2}{243|a^3|} \right), \\
& 4|a| \sqrt{\tfrac{\mathrm{M}}{3b^3}} + \tfrac{32\mathrm{M}}{9b} - \tfrac{80}{27|a|} \sqrt{\tfrac{\mathrm{M}^3}{b}} + \tfrac{832\mathrm{M}^2}{243a^2}, \\
& - \operatorname{sign}(a) \left(2a^2 \sqrt{\tfrac{\mathrm{M}}{3b^5}} + \tfrac{28|a|\mathrm{M}}{9b^2} + \tfrac{8(6\sqrt{3}-5)}{27} \sqrt{\tfrac{\mathrm{M}^3}{b^3}} + \tfrac{80(16-3\sqrt{3})\mathrm{M}^2}{243|a|b} \right), \\
& 0 \quad \Bigg) \quad + O(|\mathrm{M}|^{5/2}).
\end{aligned}
$$

3.4.2 Pictures

The surface on which H^μ lives is the level set $2H_2 = \lambda$, which is a 3-torus in $\mathbb{R}^4$. Using the $\mathbb{S}^1$-symmetry to divide out one dimension, we can make 2-dimensional pictures of level sets of H^μ on this surface. There are two natural ways of choosing coordinates, which are both singular but have their singularities in different locations on the 3-torus. We give pictures for both sets of coordinates.

The bifurcation diagram consists of the parabola (3.15) of pitchfork bifurcations, and the cubic (3.18) of saddle–node bifurcations. For $a = 1/4$ and $b = 1/3$ these curves are shown in Fig. 3.4. Note that around the origin, the bifurcation diagram is similar to Fig. 2.5.

In Fig. 3.5 the level curves of H^μ are plotted, for fixed $2H_2 = \lambda$ with $\lambda > \frac{a^2}{4b^2}$, so that a saddle–node bifurcation is expected. The two sets of pictures correspond to the Poincaré sections $\phi_2 = 0$ and $\phi_1 = 0$, corresponding to the pendulum and

the spring having zero velocity, respectively. The fundamental invariants ρ_1, ρ_2, ψ in Cartesian coordinates on either Poincaré section are

	$\rho_1:$	$\rho_2:$	$\psi:$
$\phi_2 = 0:$	$x^2 + y^2$	$4\lambda - 2(x^2 + y^2)$	$2x(2\lambda - x^2 - y^2)$
$\phi_1 = 0:$	$2\lambda - \frac{1}{2}(x^2 + y^2)$	$x^2 + y^2$	$(x^2 - y^2)\sqrt{2\lambda - (x^2 + y^2)/2}$

Note that the short periodic orbit $\rho_2 = 0$ corresponds to the origin in the section $\phi_1 = 0$ (the right-hand column of pictures), whereas it corresponds to the outer circle $x^2 + y^2 = 2\lambda$ in the section $\phi_2 = 0$. Similarly, the origin in the left-hand column, i.e., $\rho_2 = 0$, corresponds to the outer circle $x^2 + y^2 = 4\lambda$ of the other.

Besides the saddle–node and pitchfork bifurcations found in section 3.4.1, a fourth 'bifurcation' is observed. It is caused by the coincidence of the level through $\rho_1 = 0$ and the critical value of the saddle emanating from the saddle–node bifurcation. Since the subset $\rho_1 = 0$ (spring not oscillating) is not an invariant subset, this coincidence appears to be a bifurcation only because of the choice of coordinates in the right-hand column, where $\rho_1 = 0$ corresponds to the outer circle. This should be contrasted to the situation in the left-hand column, where the outer circle corresponds to $\rho_2 = 0$ (pendulum not oscillating), which *is* an invariant subset.

3.4.3 Inducing the system from the model

The Hamiltonian equation for the spring-pendulum system is derived in section 2.3.1. For reference we quote equation (2.7) up to order 4:

$$H^0(x,y) := \frac{x_1^2 + y_1^2}{2} + a_1\frac{x_2^2 + y_2^2}{2} - 4a_2x_2y_1y_2 - 8a_3x_2^4 + 8a_4x_2^2y_1^2 + 8a_5x_2^2y_2^2 + O(|x_i, y_i|^5).$$

After Birkhoff normalization around 1 : 2 resonance, the Hamiltonian can be written in terms of the fundamental invariants ρ_1, ρ_2, ψ as in equation (3.10), analogous to H^n in Sect. 2.3.2. The leading part, up to fourth order in phase variables, of equation (3.10), is

$$H^n = d_1\rho_1 + d_2\rho_2 + d_3\psi + d_4\rho_1^2 + d_5\rho_1\rho_2 + d_7\rho_2^2 + O(|x_i, y_i|^5), \tag{3.20}$$

with coefficients depending on those of the original Hamiltonian as

$$\begin{aligned} &d_1 = \frac{1}{2}, \quad d_2 = \frac{a_1}{2}, \quad d_3 = -2a_2, \quad d_4 = 0, \\ &d_5 = \frac{4(-2a_2^2 + a_4 + 2a_1a_4)}{1 + 2a_1}, \quad d_7 = \frac{-2(a_2^2 + (1 + 2a_1)(3a_3 - a_5))}{1 + 2a_1}. \end{aligned} \tag{3.21}$$

Generally, the Birkhoff normalized system depends on χ. However, since our system has a time-reversibility symmetry under which χ is sent to $-\chi$, the

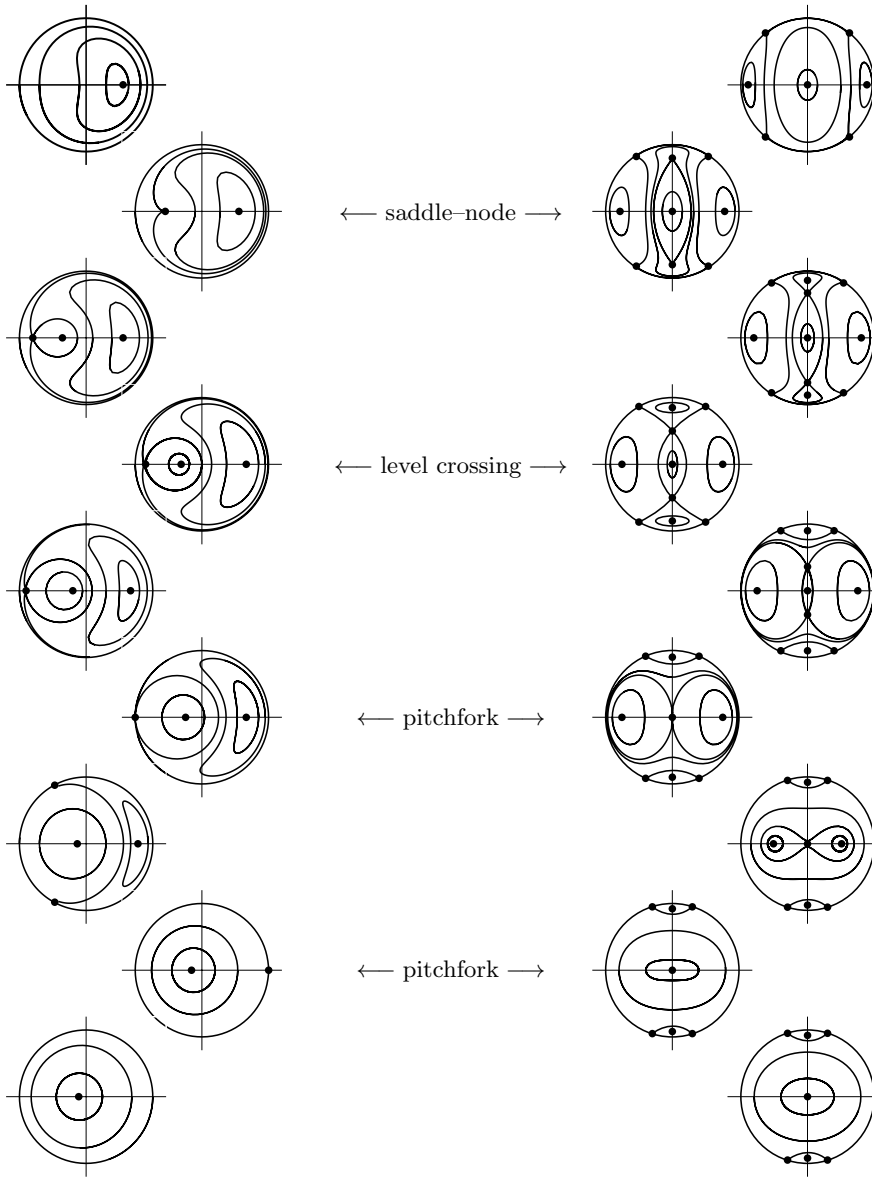


Fig. 3.5 Bifurcations of the Energy–Momentum map normal form around the 1:2 resonance (large λ). The left-hand column corresponds to the section $\phi_2 = 0$ with a return time $\approx 4\pi$; the section $\phi_1 = 0$ (right-hand column) has a return time $\approx 2\pi$.

normalized Hamiltonian only depends on even powers of χ. By using the relation $\psi^2+\chi^2 = \rho_1\rho_2^2$, all of these terms can be removed. Hence, a second normalization to remove terms involving χ is not necessary.

In the new variables, the conserved quantity is $\lambda := \rho_1 + \frac{1}{2}\rho_2$. Pulling this back to old coordinates yields

$$\lambda = \left(x_1^2 + y_1^2\right) + \frac{1}{2}\left(x_2^2 + y_2^2\right) + \frac{8a_2}{1+2a_1}\left(-x_1x_2^2 + 2x_2y_1y_2 + x_1y_2^2\right) + O(|x_i, y_i|^4).$$

(For λ up to fourth order terms, see (2.8).)

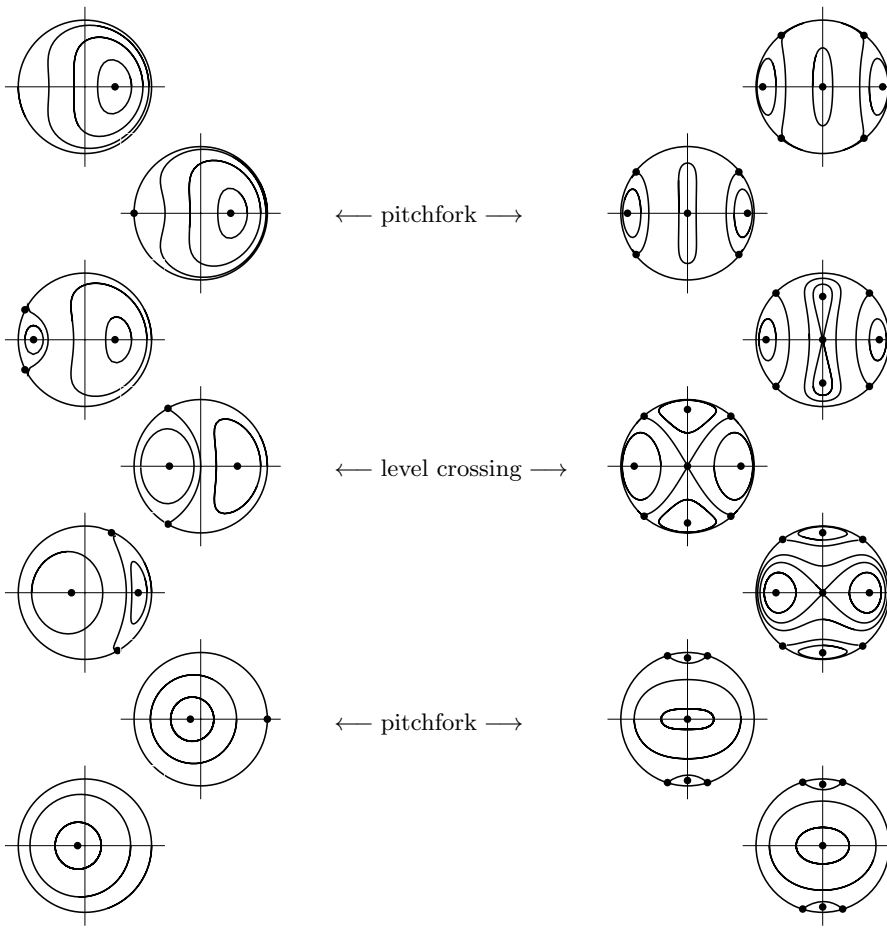


Fig. 3.6 Bifurcations of the Energy–Momentum map normal form around the 1:2 resonance (small λ).

Birkhoff nondegeneracy conditions Depending on the order to which Birkhoff normalization is performed, a number of nondegeneracy conditions are encountered. From the expressions for d_5 and d_7 the condition $a_1 \neq -1/2$ follows

immediately. If normalization is continued up to eighth order, we should exclude the following values for a_1:

$$a_1 \neq -\frac{3}{2},\ a_1 \neq -1,\ a_1 \neq -\frac{1}{2},\ a_1 \neq -\frac{1}{4},\ a_1 \neq 0,\ a_1 \neq \frac{1}{4},\ a_1 \neq 1,\ a_1 \neq \frac{3}{2}.$$

(We remark that as the normalization order is increased, more and more rational values of a_1 must be excluded. In the limit of infinite order, all rational numbers[3], except the value $\frac{1}{2}$ around which we normalize, should be excluded. This is related to the non-convergence of the normalizing formal power series. Effectively, this means that the higher the order to which normalization is performed, the smaller is the allowed deviation of a_1 from the resonant value.)

Energy–momentum map nondegeneracy conditions In Sect. 3.3.1 the tangent space to an arbitrary Hamiltonian (3.20) is computed, assuming that it is Birkhoff normalized around the 1 : 2 resonance. This tangent space has minimal codimension (namely 1) if two nondegeneracy conditions are met. We may pull back these to the space of original parameters, simply by substituting the expressions for the coefficients d_i. This yields the following nondegeneracy conditions:

$$\begin{aligned} &0 \neq a_2, \\ &0 \neq 4(1+4a_1)a_2^2 - (1+2a_1)(6a_3 - a_4 - 2a_5). \end{aligned}$$

Normalizing transformation From now on we suppose that the nondegeneracy conditions hold. We may then compute a standard basis for the left-right tangent space of H^n. This serves as input for Kas and Schlessinger's algorithm, with which we can compute transformations that connect H^n to the versal unfolding H^u obtained in Sect. 3.3.1, namely

$$H^u_\mu = \frac{1}{2}\rho_1 + \left(\frac{1}{4} + \mu\right)\rho_2 + a\psi + b\rho_1\rho_2. \tag{3.22}$$

This model has the drawback that generically a Hamiltonian H^n cannot be induced from it by transformations that are the identity to first order. Theoretically this is no problem. An practical nuisance however is that for such models, Kas and Schlessinger's algorithm does not yield a finite part of the formal power series in a finite number of steps.

It turns out that adding a second deformation direction, for instance $c\rho_1^2$, solves this problem. For reasons of convenience, and while it does not complicate the analysis, we chose to add *two* extra deformation directions, and use the model

$$H^U_\mu = \frac{1}{2}\rho_1 + \left(\frac{1}{4} + \mu\right)\rho_2 + a\psi + b\rho_1^2 + c\rho_1\rho_2 + d\rho_2^2. \tag{3.23}$$

[3] modulo a scaling by a factor 2

By following the calculation of Sect. 3.4.1 we then arrive at this equation for the pitchfork bifurcation:

$$(\mu + (b-c)\lambda)^2 - a^2\lambda = 0,$$

which occurs at

$$(r_1, r_2, \psi, \chi) = \left(\left|\frac{1}{a}\mu + \frac{b-c}{a^3}\mu^2 + \frac{2(b-c)^2}{a^5}\mu^3\right|, 0, 0, 0\right) + O(\mu^4).$$

Now we discuss the computation of the inducing transformation. Since the unfolding H^U has fewer terms than the target Hamiltonian H^n, it is more efficient to compute the inverse transformation, going from H^U to H^n. In particular, the expressions for the standard basis of H^U's tangent space are much cleaner than H^n's.

Let $\mathbb{R}^d \ni (a_1, a_2, \ldots)$ be the space of parameters of the Birkhoff-normalized system H^n. We computed functions

$$\begin{aligned} h(a_i) &: \mathbb{R}^p \to \mathbb{R}, \\ A(\rho_1, \rho_2, \psi; a_i) &: \mathbb{R}^3 \oplus \mathbb{R}^d \to \mathbb{R}^3, \\ B(y_1, y_2; a_i) &: \mathbb{R}^2 \oplus \mathbb{R}^d \to \mathbb{R}^2 \end{aligned}$$

such that these induce the actual system H^n from the versal unfolding H^U_μ:

$$H^n \quad = \quad B \circ (H^U_{h(\mu)}, H_2) \circ A.$$

These transformations depend on the parameters a, b, c, d of the versal model (3.22). Several choices of these parameters yield transformations that are the identity to first order, which is related to there being one deformation direction too much for this purpose. The natural choice is of course

$$a = d_3 = -2a_2, \qquad b = d_4 = 0, \qquad c = d_5, \qquad d = d_7,$$

see also (3.21). With this choice, the resulting transformations are

$$\begin{aligned} h = \quad & -\frac{1}{4}(1 - 2a_1) \\ A_1(\rho_1, \rho_2, \psi) = \quad & \rho_1 + \left(\frac{16a_2^2}{(1+2a_1)^2} - \frac{8(4a_3 + a_4 + 4a_4)}{3(1+2a_1)} - \frac{4a_6}{3a_2}\right)\rho_1\rho_2 + \\ & \left(\frac{-192a_2^3}{(1+2a_1)^2} + \frac{32a_2(8a_3 + 5a_4 + 8a_5)}{3(1+2a_1)} + \frac{32a_6}{3}\right)\psi\rho_1 + h.o.t. \\ A_2(\rho_1, \rho_2, \psi) = \quad & \rho_2 + \left(\frac{-32a_2^2}{(1+2a_1)^2} + \frac{16(4a_3 + a_4 + 4a_4)}{3(1+2a_1)} + \frac{8a_6}{3a_2}\right)\rho_1\rho_2 + \\ & \left(\frac{-192a_2^3}{(1+2a_1)^2} + \frac{32a_2(8a_3 + 5a_4 + 8a_5)}{3(1+2a_1)} + \frac{32a_6}{3}\right)\psi\rho_1 + h.o.t. \\ A_3(\rho_1, \rho_2, \psi) = \quad & \psi + \left(\frac{-288a_2^3}{(1+2a_1)^2} + \frac{16a_2(8a_3 + 5a_4 + 8a_5)}{(1+2a_1)} + 16a_6\right)\psi^2 + \\ & \left(\frac{-8a_2^2}{(1+2a_1)^2} + \frac{4(4a_3 + a_4 + 4a_4)}{3(1+2a_1)} + \frac{2a_6}{3a_2}\right)(4\psi\rho_1 - \psi\rho_2) + h.o.t. \end{aligned}$$

$$
\begin{aligned}
B_1(y_1, y_2) = \quad & y_1 + O(|y_1, y_2|^3 + |a_i|^3) \\
B_2(y_1, y_2) = \quad & y_2 + \left(\frac{-24a_2^2}{(1+2a_1)^2} + \frac{4(8a_3 + 5a_4 + 8a_5)}{3(1+2a_1)} + \frac{4a_6}{3a_2} \right) (4y_1y_2 - y_2^2) + \\
& + O(|y_1, y_2|^3 + |a_i|^3)
\end{aligned}
$$

Here *h.o.t.* denote terms of order n or higher in the phase variables, and order 3 or higher in the parameters a_i.

It remains to find the bifurcation curves, in terms of the detuning parameter $1-2a_1$ and the integral λ. Because the transformations above are not symplectic, the λ-level sets of H^U transformed by this coordinate transformation are not dynamically invariant, and do not correspond to λ-level sets of H^n. Hence, to find the bifurcation curves, we need to find the location of bifurcations in H^U, transform it back to original variables via A^{-1}, and compute λ at those phase points. Since A is the identity to first order, its inverse is easy to find. The pitchfork bifurcation in H^U occurs on the line $\rho_2 = \psi = 0$, and since A does not involve ρ_1^2 (in fact, it does not include ρ_1^3 terms either) the inverse also does not contain these terms. Hence, on line $\rho_2 = \psi = 0$ the map A^{-1} is the identity to at least third order.

Calculating λ at the point (3.4.3) and expanding in $1 - 2a_1$ results in the following expression for the pitchfork bifurcation:

(3.24)
$$
\lambda = \frac{(1-2a_1)^2}{64a_2^2} + \frac{(a_2^2 - a_4)(1-2a_1)^3}{128a_2^4} + \frac{(a_2^4 - 10a_2^2a_4 + 5a_4^2)(1-2a_1)^4}{1024a_2^6} + h.o.t.,
$$

where the *h.o.t.* stand for $O((1-2a_1)^5)$ terms. It agrees with (2.14) found using the planar reduction method.

For the saddle–node bifurcation it is not possible to find a similar equation. The reason is that this bifurcation occurs at a finite distance from the origin, and only for $O(1)$ values of the detuning parameter $1 - 2a_1$, see (3.19), and the transformation A and its inverse are only valid near the origin.

Table 3.1 Comparison of bifurcation values, found numerically and analytically. Error bounds in last digit(s) of measured quantities are given

H	a_1	a_2	a_3	a_4	a_5	$\lambda_{\text{measured}}$	$\lambda_{\text{predicted}}$
.004	$.47513 \pm 1$	.1	0	$-.3$	.2	$.008025 \pm 5$	$--$
.004	$.48405 \pm 5$	.1	0	$-.3$	.2	$.008000 \pm 2$	$.00286 \pm 2$
.004	$.5562 \quad \pm 2$	.1	0	$-.3$	.2	$.007999 \pm 1$	$.060 \pm 1$
.004	$.46454 \pm 1$	.1	0	$-.01$	.35	$.008000 \pm 4$	$--$
.004	$.46550 \pm 1$	.1	0	$-.01$	.35	$.008000 \pm 3$	$.007988 \pm 5$
.004	$.53705 \pm 1$	.1	0	$-.01$	.35	$.007999 \pm 1$	$.007991 \pm 4$
.0004	$.48882 \pm 1$	.1	0	$-.01$	.05	$.00080000 \pm 1$	$.0007991 \pm 14$
.0004	$.51144 \pm 1$	.1	0	$-.01$	.05	$.00080000 \pm 1$	$.0007997 \pm 13$

Numerical Poincaré sections To validate (3.24) we numerically computed an iso-energy Poincaré map, using the section $y_2 = 0$ and increasing; i.e., transversal to the long periodic orbit. We fixed the energy, and coefficients $a_2, a_3, \ldots$, while varying a_1 to find the bifurcation value. The value of λ was then measured along trajectories close to the bifurcation point.

Error bounds in a_1 correspond to values that bracket the actual bifurcation value. These bracketing values were determined by computing phase diagrams and inspecting them visually. The error in $\lambda_{\text{predicted}}$ was obtained by propagating the errors in a_1. Error bounds in $\lambda_{\text{measured}}$ correspond to the observed variation in the computed value for λ along a trajectory. This error bound may be too low, as it obviously depends on the trajectory. An alternative estimation method, estimating the remainder in the series (2.8) for λ, seems difficult.

Another perturbing effect, which influences the error bound in $\lambda_{\text{measured}}$, is the effect of *chaotic* dynamics, that is, the effect of the higher-order perturbation terms that are thrown away in order to arrive at an integrable model. For the energies considered, this effect is extremely small. However, this is a statement of belief rather than a hard fact. It is support by the fact that the nonintegrable effects are due to a flat perturbation, and that the chaotic regime begins to have a 'noticeable' effect only for energies a factor 10 to 100 higher than we consider.

The agreement of the predicted bifurcation value of λ and its actual value is excellent for small values of a_4 and small values of the energy. For the case $a_4 = -0.3$, the three consecutive terms in (3.24) are of similar magnitude, which suggests that more terms are needed for convergence, in agreement with the large deviation of $\lambda_{\text{predicted}}$ from the observed value of λ.

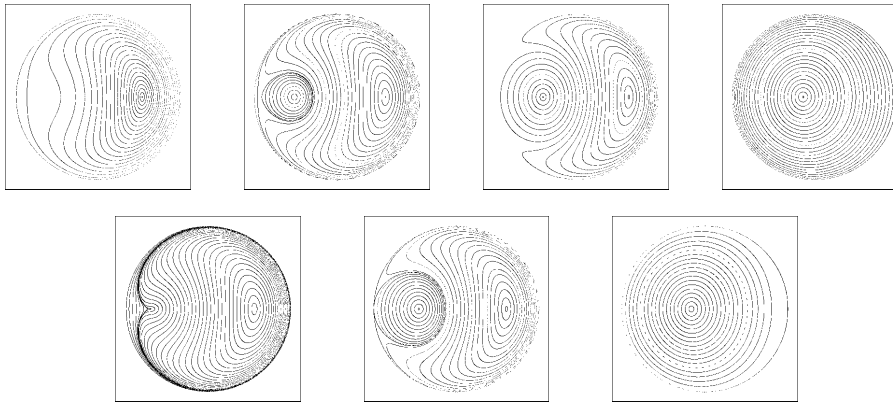
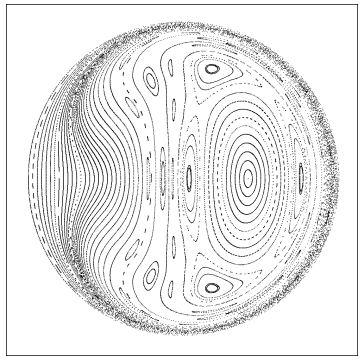
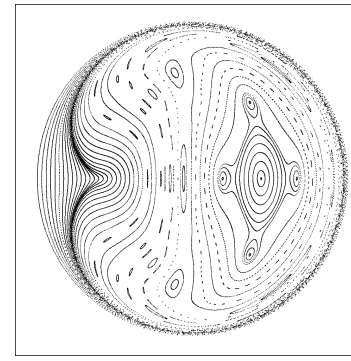


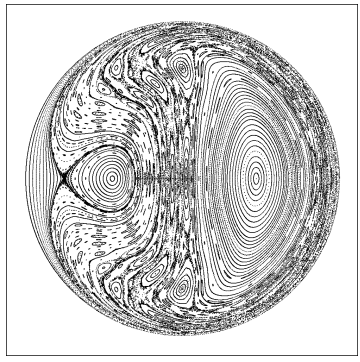
Fig. 3.7 Bifurcations in the Poincaré section around the 1:2 resonance, for $H = 0.004$, $a_2 = 0.1$, $a_3 = 0$, $a_4 = -0.3$, $a_5 = 0.2$, other coefficients 0, and a_1 increasing from left to right: 0.47,0.47513, 0.48, 0.48405, 0.49, 0.5562, 0.57.



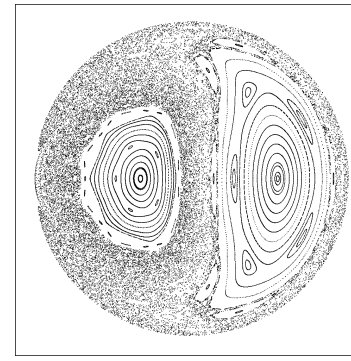
$a_1 = 0.46$



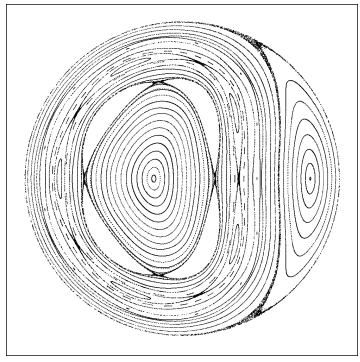
$a_1 = 0.47$



$a_1 = 0.48$



$a_1 = 0.51$



$a_1 = 0.60$

Fig. 3.8 Some large-energy Poincaré sections: $H = 0.1$. Chaotic regimes and subharmonic bifurcations are clearly visible, with the backbone of Fig. 3.7 still present.

4 Birkhoff normalization

The Birkhoff normal form procedure is a widely used tool for approximating a Hamiltonian systems by a simpler one. This chapter starts out with an introduction to Hamiltonian mechanics, followed by an explanation of the Birkhoff normal form procedure. Finally we discuss several algorithms for computing the normal form.

4.1 Introduction

Hamiltonian systems with more than 1 degree of freedom are difficult to analyze directly.[1]. The Birkhoff procedure is an iterative procedure for constructing a coordinate transformation that normalizes the system, so that it has extra conserved quantities or *integrals*, and associated *symmetries*. Using these, the system can be reduced to fewer degrees of freedom, see e.g. [AM78, CS85], making the analysis more feasible.

The coordinate transformation resulting from the Birkhoff procedure is a *formal* transformation, which need not converge to an analytic function. The formal series can always be lifted to a $\mathbb{C}^\infty$ transformation, but generically there does not exist a lifting that actually conjugates the original system and the normalized one, e.g. see [BT89]. This ties in with the fact that Hamiltonian systems with at least two degrees of freedom generically are nonintegrable, whereas Birkhoff normalized systems do; see also [Ito89].

Hence, the symmetries and integrals obtained are only *approximate*, up to flat perturbations. For small excitations these perturbations are extremely small, so that integral curves of the system stay close to those of the normalized system for a long time. For two degrees of freedom systems even more is true: KAM tori then prevent the occurrence of Arnol'd diffusion, so that solution curves stay close to the integrable system's tori for all time; see also the introduction to Chap. 2.

Birkhoff's original result [Bir50] dealt with Hamiltonians with a so-called nonresonant quadratic part, close to an elliptic equilibrium. For such a system on, say, a $2n$ dimensional phase space, Birkhoff's method yields $2n$ new coordinates, of which n are integrals of motion, and n are associated *cyclic variables*, each

[1] But see [Sim96].

H. Broer, I. Hoveijn, G. Lunter, and G. Vegter: LNM 1806, pp. 71–84, 2003.

living on a 1-sphere. The system is independent of these n variables, i.e., the system has a $\mathbb{T}^n$ torus symmetry.

Birkhoff's ideas have since been generalized considerably. It is used in the neighborhood of non-elliptic equilibria, around periodic trajectories or invariant tori, and also at resonances. In this work, we are interested in *resonant* Hamiltonians close to *equilibrium*. In this more general situation, Birkhoff's procedure still yields integrals, albeit not as many as before. This is the situation in which we apply the reduction methods of Chaps. 2 and 3; see the respective introductions for details.

This chapter is organized as follows. The first section introduces some aspects of Hamiltonian mechanics, the most important one being the exponential formula for coordinate changes. Then we state and prove Birkhoff's result in the versions we use in Chaps. 2 and 3. The last part is devoted to algorithms for computing the Birkhoff normal form.

4.2 Introduction to Hamiltonian mechanics

The aim of this section is to introduce the notions required to state and prove the Birkhoff normal form theorem, and also to introduce the mathematics of Hamiltonian mechanics. For more thorough treatments see [Arn89, MH92, Sja90].

In Hamiltonian mechanics there is an isomorphism[2] between (Hamiltonian) vector fields and the corresponding Hamiltonian function, or *Hamiltonian*, H. This isomorphism is given by the *symplectic structure*. Both the the Hamiltonian function and the associated vector field may be transformed by changes of coordinates. The *symplectic* coordinate transformations are, by definition, those that respect the isomorphism. This means that within the class of symplectic transformations, computations on the level of vector fields can also be done on the level of the Hamiltonian functions, which is much easier.

Without much motivation, we now give the basics of Hamiltonian mechanics, and refer to [Arn89, MH92, Sja90] for more details. Let $R = \mathbb{R}^{2n}$ be an even-dimensional space. The coordinates on this space come in pairs $p_1, q_1, \ldots, p_n, q_n$, where the variable p_i is called the *momentum variable* conjugate to the *configuration variable* q_i. Heuristically, q_i is a position coordinate, and p_i the associated momentum or 'velocity'. Let H be a function on R. Associated to it is a vector field $\boldsymbol{X}_H$ defined by

$$\boldsymbol{X}_H = \sum_i \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}. \tag{4.1}$$

Here $\frac{\partial}{\partial q_i}$ and $\frac{\partial}{\partial p_i}$ are the constant vector fields in the q_i and p_i-directions, respectively. The time-t flow of this vector field is denoted by $\boldsymbol{X}_H^t$. For the derivative of an arbitrary function F in the direction of $\boldsymbol{X}_H$ we write $\boldsymbol{X}_H F$ or $\{H, F\}$:

[2] This holds except for possible global obstructions; on $\mathbb{R}^{2n}$ there are none.

$$\{H, F\} \;:=\; \boldsymbol{X}_H F \;:=\; \frac{\partial}{\partial t} F \circ \boldsymbol{X}_H^t|_{t=0} \;=\; \sum_i \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial F}{\partial p_i}.$$

The function $\{H, F\}$ is called the *Poisson-bracket* of H and F. It is bilinear and antisymmetric, satisfies the Jacobi-identity, and is isomorphic to the ordinary Lie-bracket on vector fields via $\boldsymbol{X}$, that is:

$$\boldsymbol{X}_{\{H,F\}} = [\boldsymbol{X}_H, \boldsymbol{X}_F]. \tag{4.2}$$

Since applying the Poisson-bracket to homogeneous elements yields such elements again, (4.2) induces the structure of a *graded Lie algebra* [Bro79, Bro81] when the base ring and the vector field module are 'chopped up' into homogeneous parts. This structure will be used later.

Definition (4.1) of $\boldsymbol{X}_H$ depends on the coordinates p_i and q_i, which is unpleasant when transforming coordinates. For this reason, $\boldsymbol{X}_H$ is usually defined in a coordinate-free manner as the unique vector field satisfying

$$\mathrm{d}H = \omega(\cdot, \boldsymbol{X}_H),$$

where ω is a nondegenerate differential 2-form, called the *symplectic form*. In the coordinates used above it assumes the form $\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i$. Conversely, any nondegenerate closed 2-form ω can locally be written in this way; this is Darboux' theorem. Coordinate transformations with respect to which ω is equivariant are called *symplectic*, and also leave the Hamiltonian differential equations (4.1) invariant.

It is easy to see that the flow $\boldsymbol{X}_F^t$ of an arbitrary Hamiltonian F is a coordinate transformation; indeed, $\boldsymbol{X}_F^{-t}$ is its inverse. It turns out that such transformations are symplectic, see e.g. [Arn89, p. 204]. It is therefore natural to try to simplify a Hamiltonian system H by conjugating it with a symplectic transformation generated by another Hamiltonian F. The function $H \circ \boldsymbol{X}_F^t$, as a function of t, satisfies the differential equation

$$\frac{\partial}{\partial t} H \circ \boldsymbol{X}_F^t = \{F, H \circ \boldsymbol{X}_F^t\}. \tag{4.3}$$

We now define ad_F to be the operator $\mathrm{ad}_F : H \mapsto \{F, H\}$. This is a linear operator, whence the solution to (4.3) is

$$H \circ \boldsymbol{X}_F^t = \exp(t\,\mathrm{ad}_F)H := H + t\,\mathrm{ad}_F(H) + t^2\frac{1}{2!}\,\mathrm{ad}_F(\mathrm{ad}_F(H)) + \cdots \tag{4.4}$$

(See also [Grö67]). This formula is behind the proof of the Birkhoff normal form theorem.

Remark 4.1. *(Formal series)* We denote by $\mathcal{H}_i \subseteq R$ the subspace consisting of homogeneous elements of degree i. If F has no linear part, then ad_F maps $\mathcal{H}_i$ into $\mathcal{H}_i \oplus \mathcal{H}_{i+1} \oplus \cdots$, implying that (4.4) converges to a formal power series. (If F has a linear part, then $\boldsymbol{X}_F^t$ corresponds to a shift of the origin, and conjugation of formal power series with such transformations makes no sense.)

4.3 Birkhoff normal form theorem

The idea of Birkhoff's normal form procedure is to conjugate a Hamiltonian system H with the flow of another Hamiltonian F, in order to simplify it. It is an iterative procedure, normalizing the system degree by degree. The end result is a Hamiltonian H' in normal form, which has a circle or torus symmetry, and associated conserved quantities.

The Birkhoff procedure results in a formal power series, and any C^∞ transformation with this power series as Taylor series forms a conjugation between H and H', modulo a flat perturbation; see the introduction for more remarks. Although our interest is in practical computations, we below give Birkhoff's result in the formal power series setting, which allows for a clear formulation.

We restrict our attention to the case of normalizing around an equilibrium, which amounts to requiring that the Hamiltonian has vanishing linear part. In that case, the change in a Hamiltonian system H, when subjected to a coordinate change generated by F is to first order described by $\operatorname{ad}_{H_2}(F)$. Here H_2 is the quadratic part of H, and ad_{H_2} is the associated *adjoint operator*. It is just the derivative of F along the vector field associated to H_2. The image of ad_{H_2} describes, to great extent, the normal form to which H can be transformed. In particular, if ad_{H_2} is semisimple, then $\ker \operatorname{ad}_{H_2}$ complements $\operatorname{im} \operatorname{ad}_{H_2}$ and, if F is chosen appropriately, the transformed system H' will lie in the kernel of ad_{H_2}. (The equation that F has to satisfy is called the *adjoint equation*). This means that H' is constant under the flow of H_2, that is, the flow of H_2 is a symmetry of H'. Conversely, since $\operatorname{ad}_{H'}(H_2) = -\operatorname{ad}_{H_2}(H') = 0$, we see that H_2 is constant under the flow of H'; indeed, H_2 is the conserved quantity associated to the symmetry. If often happens that more functionally independent quantities are in $\ker \operatorname{ad}_{H_2}$, leading to more symmetries; see Corollary 4.3.

Theorem 4.2. *Let H be a Hamiltonian in $\mathbb{R}[[\underline{x}]]$ without linear part, and write it as*

$$H = H_2 + H_3 + \cdots$$

where $H_i \in \mathcal{H}_i$ is the homogeneous part of degree i; see remark 4.1. Let $\mathcal{G}_i \subseteq \mathcal{H}_i$ be linear subspaces such that $\mathcal{G}_i + \operatorname{Im} \operatorname{ad}_{H_2} = \mathcal{H}_i$. Then there exists a formal symplectic power series transformation Φ such that

$$H \circ \Phi = H_2 + \widetilde{H}_3 + \widetilde{H}_4 + \cdots$$

where $\widetilde{H}_i \in \mathcal{G}_i$ $(i = 3, 4, \ldots)$.

Proof: [Tak74c, Arn89, Bro79, Bro81, Mee85, MH92] The proof is by induction. Assume ϕ_i normalizes H up to order i, so that the partly normalized Hamiltonian is of the form

$$H \circ \phi_i = H_2 + \widetilde{H}_3 + \cdots + \widetilde{H}_i + H'_{i+1} + \cdots$$

where $\widetilde{H}_j \in \mathcal{G}_j$, $j = 3, \ldots, i$. Choose $F_{i+1} \in \mathcal{H}_{i+1}$ such that $\widetilde{H}_{i+1} := H'_{i+1} - \operatorname{ad}_{H_2}(F_{i+1}) \in \mathcal{G}_{i+1}$, and define $\phi_{i+1} := \phi_i \circ \boldsymbol{X}^1_{F_{i+1}}$. By (4.4) we get

$$\begin{aligned} H \circ \phi_{i+1} &= H_2 + \widetilde{H}_3 + \cdots + \widetilde{H}_i + H'_{i+1} + \cdots + \mathrm{ad}_{F_{i+1}}(H_2) + \mathrm{ad}_{F_{i+1}}(\widetilde{H}_3) + \cdots \\ &= H_2 + \widetilde{H}_3 + \cdots + \widetilde{H}_i + (H'_{i+1} - \mathrm{ad}_{H_2}(F_{i+1})) + \cdots \\ &= H_2 + \widetilde{H}_3 + \cdots + \widetilde{H}_i + \widetilde{H}_{i+1} + H''_{i+2} + \cdots . \end{aligned}$$

(Here we used that ad_{H_2} maps $\mathcal{H}_i$ to itself.) So ϕ_{i+1} normalized H up to order $i+1$. Since ϕ_i and ϕ_{i+1} coincide up to terms of order i, the sequence $(\phi_i)_{i=1}^{\infty}$ converges as a formal power series. ∎

To see how theorem 4.2 implies the extra symmetries, we also formulate Birkhoff's original result:

Corollary 4.3. *Let $H \in \mathbb{R}[[\underline{x}]]$ be a Hamiltonian without linear part, and with quadratic part*

$$H_2 = \omega_1(q_1^2 + p_1^2) + \cdots + \omega_n(q_n^2 + p_n^2).$$

Assume that the non-resonance condition holds: For all nonzero vectors $(k_1, \ldots, k_n)$ in $\mathbb{Z}^n$ we have $k_1\omega_1 + \cdots + k_n\omega_n \neq 0$. Then there exists a formal symplectic power series transformation Φ such that

$$H \circ \Phi = H'(L_1, \ldots, L_n),$$

where $L_i = q_i^2 + p_i^2$. The system H' has a $\mathbb{T}^n$ torus symmetry.

The L_i are the 'radial' part of so-called symplectic Hamiltonian polar coordinates (L_i, ϕ_i). The system H' is independent of the ϕ_i, so that the $\mathbb{T}^n$ torus action

$$\zeta : \quad \mathbb{T}^n \times R \to R : \quad (\rho_i, L_i, \phi_i) \mapsto (L_i, \phi_i + \rho_i)$$

is a $\mathbb{T}^n$-symmetry of the system. The L_i are conserved by the flow, since by Hamilton's equations $\frac{\partial}{\partial t} L_i = \frac{\partial H}{\partial \phi_i} = 0$.

Proof of the corollary: It is convenient to use complex coordinates $z_i = q_i + \mathrm{i}p_i$, then $H_2 = \omega_1 z_1 \bar{z}_1 + \cdots$, and it is easily checked that monomials in these variables are eigenfunctions of the adjoint operator:

$$\mathrm{ad}_{H_2}(z_1^{\alpha_1}\bar{z}_1^{\beta_1}\cdots z_n^{\alpha_n}\bar{z}_n^{\beta_n}) = (\omega_1(\alpha_1 - \beta_1) + \cdots + \omega_n(\alpha_n - \beta_n))\,(z_1^{\alpha_1}\bar{z}_1^{\beta_1}\cdots z_n^{\alpha_n}\bar{z}_n^{\beta_n}).$$

Using the non-resonance condition, it is seen that the eigenvalues are zero exactly for elements of the algebra generated by the functions $L_1 = z_1\bar{z}_1, \ldots, L_n = z_n\bar{z}_n$. Taking $\mathcal{G}_i$ to be the subspace of degree-i elements of this algebra, the result follows directly from the conclusion of theorem 4.2. ∎

Remark 4.4. *(Symmetric normalizing)* Often the Hamiltonian H is invariant with respect to some symmetry group Γ. We assume that Γ is compact and acts linearly on the phase space. We also assume that Γ respects the symplectic structure, i.e., $\{F, G\} \circ \gamma = \{F \circ \gamma, G \circ \gamma\}$ for all $\gamma \in \Gamma$. Then, in theorem 4.2 we may restrict to the space of Γ-invariant Hamiltonians, and ϕ can be chosen to

be Γ-equivariant, as the image of ad_{H_2} restricted to Γ-invariant Hamiltonians is element-wise Γ-invariant. If Γ *anti*-commutes with the Poisson-bracket, as happens if Γ is the time-reversal symmetry, then above remarks still hold true but the argument involves a number of minus-signs.

Remark 4.5. *(Normalizing a family)* In our application, the Hamiltonian H depends on small *parameters* u_i. In particular, the quadratic part depends on the u_i, so that $\ker \mathrm{ad}_{H_2}$ is also dependent on the parameters. However, it follows from the proof that one can find F_i, with rational functions of the u_i as coefficients, such that $\widetilde{H}_i \in \ker \mathrm{ad}_{H_2^0}$, where H_2^0 is the quadratic part of H for $u_1 = u_2 = \cdots = 0$. Moreover, the coefficients of the $\widetilde{H}_i$ are also rational functions in the u_i. It can be shown that the coefficient-values for which a term F_i or $\widetilde{H}_i$ is singular (i.e., where its coefficients have a pole) are generally dense in parameter-space, and correspond to high-order resonances in H_2. If one normalizes only to finite order, a full neighborhood of the origin is free from such singularities.

Remark 4.6. *(Generating function)* The normalizing transformation ϕ is constructed as a composition $\boldsymbol{X}^1_{F_3} \circ \boldsymbol{X}^1_{F_4} \circ \cdots$, with infinitely many components $\boldsymbol{X}^1_{F_i}$ and $F_i \in \mathcal{H}_i$. The resulting transformation may be written in the form $\boldsymbol{X}^1_{F'}$, where $F' = F'_3 + F'_4 + \cdots$, and again $F'_i \in \mathcal{H}_i$. The relation between the F_i and the F'_i is given by the Campbell-Baker-Hausdorff formula; see [DF76].

4.3.1 Semisimple quadratic part, and resonance

In this section, Theorem 4.2 is applied to the case where H has a semisimple quadratic part without or with just one resonance.

Instead of coordinates p_i, q_i we use complex $\mathbb{R}$-linearly independent coordinates $z_i, \bar{z}_i$ defined by $z_i := q_i + \mathrm{i}p_i$, $\bar{z}_i := q_i - \mathrm{i}p_i$. By assumption H_2 is semisimple, and in these coordinates this means that the matrix X_{H_2} is diagonalizable. Transforming coordinates (symplectically), we may suppose that H_2 is of the form[3]

$$H_2 = \mathrm{i}\omega_1 z_1 \bar{z}_1 + \cdots + \mathrm{i}\omega_n z_n \bar{z}_n. \tag{4.5}$$

In complex coordinates, $\omega = \mathrm{d}z \wedge \mathrm{d}\bar{z}$. It follows that since X_{H_2} is semisimple, so is the operator ad_{H_2}. In fact its eigenfunctions are the monomials, in complex coordinates:

$$\begin{aligned}\mathrm{ad}_{H_2}(z_1^{\alpha_1}\bar{z}_1^{\beta_1}\cdots z_n^{\alpha_n}\bar{z}_n^{\beta_n}) &= \left(\sum_{i=1}^n \omega_i\beta_i - \omega_i\alpha_i\right) z_1^{\alpha_1}\bar{z}_1^{\beta_1}\cdots z_n^{\alpha_n}\bar{z}_n^{\beta_n} \\ &= (\omega, \beta - \alpha) z_1^{\alpha_1}\bar{z}_1^{\beta_1}\cdots z_n^{\alpha_n}\bar{z}_n^{\beta_n}.\end{aligned}$$

[3] The transformation from coordinates $\{p_1, p_2, \ldots, q_1, q_2, \ldots\}$ to $\{z_1, z_2, \ldots, \bar{z}_1, \bar{z}_2, \ldots\}$ is a symplectic transformation with *multiplier* 2i. This accounts for the factor i in (4.5). See also Chap. 2, remark 2.3.

The kernel of ad_{H_2} is spanned by the monomials with exponents $(\alpha, \beta) \in \mathbb{N}^{2n}$ satisfying $(\beta - \alpha, \omega) = 0$. This set of exponents forms a semi-lattice (a set that is closed under addition, not under subtraction), therefore $\ker \mathrm{ad}_{H_2}$ is an algebra. We consider two cases, namely that the components of ω are independent over $\mathbb{Q}$ (the non-resonant case), and that there exists essentially one relation over $\mathbb{Q}$.

Lemma 4.7. *Let H_2 be of the form (4.5). Define $A := \{(\alpha, \beta) \in \mathbb{N}^{2n} : (\beta - \alpha, \omega) = 0\}$. Let a_i be the exponent vector associated to the monomial $z_i \bar{z}_i$.*

a) If the ω_i are independent over $\mathbb{Q}$, then $A = \mathbb{N}\{a_1, \ldots, a_n\}$ and $\ker \mathrm{ad}_{H_2}$ is generated, as an algebra, by

$$\{z_1 \bar{z}_1, \ldots, z_n \bar{z}_n\}.$$

b) If the ω_i obey one (up to scalar multiplication) non-trivial relation $\sum_i (\nu_i - \mu_i)\omega_i = 0$ with $\nu_i, \mu_i \in \mathbb{N}$ and $\nu_i \mu_i = 0$, then $A = \mathbb{N}\{a_1, \ldots, a_n, (\mu, \nu), (\nu, \mu)\}$ and $\ker \mathrm{ad}_{H_2}$ is generated, as an algebra, by

$$\{z_1 \bar{z}_1, \ldots, z_n \bar{z}_n, z_1^{\nu_1} \bar{z}_1^{\mu_1} \cdots z_n^{\nu_n} \bar{z}_n^{\mu_n}, z_1^{\mu_1} \bar{z}_1^{\nu_1} \cdots z_n^{\mu_n} \bar{z}_n^{\nu_n}\}.$$

The use of this lemma lies in the fact that, for semisimple operators like ad_{H_2} we have

$$\ker \mathrm{ad}_{H_2} \oplus \mathrm{Im}\, \mathrm{ad}_{H_2} = \bigoplus_{i=3}^{\infty} \mathcal{H}_i.$$

In other words, for the $\mathcal{G}_i$ in theorem 4.2 we can take $\ker \mathrm{ad}_{H_2} \cap \mathcal{H}_i$.

Remark 4.8. *(Unique normal form)* If H_2 is nonresonant, this choice of $\mathcal{G}_i$ is best possible and the normal form is unique. In case of resonance, a normal form with $\mathcal{G}_i = \ker \mathrm{ad}_{H_2}$ is not unique; see [CS85, SvdM92, Mee85]. See also Sect. 4.3.2, where this non-uniqueness is exploited in the case of the $1:2$-resonance.

More specifically, for a singly-resonant system the statement is like this:

Proposition 4.9. *Let H be a Hamiltonian with n degrees of freedom. Suppose that the quadratic part H_2 is of the form*

$$H_2 = \mathrm{i}\omega_1 z_1 \bar{z}_1 + \mathrm{i}\omega_2 z_2 \bar{z}_2 + \cdots + \mathrm{i}\omega_n z_n \bar{z}_n$$

with $\frac{\omega_2}{\omega_1} = \frac{p}{q}$, $\gcd(p, q) = 1$, $q > 0$, and suppose there are no further relations over $\mathbb{Q}$ between the ω_i. Then there exists a formal symplectic coordinate transformation ϕ such that

$$H \circ \phi = \begin{cases} H_2 + f(z_1 \bar{z}_1, \cdots, z_n \bar{z}_n, z_1^p \bar{z}_2^q, \bar{z}_1^p z_2^q) & \text{if } p > 0, \\ H_2 + f(z_1 \bar{z}_1, \cdots, z_n \bar{z}_n, z_1^{-p} z_2^q, \bar{z}_1^{-p} \bar{z}_2^q) & \text{if } p < 0. \end{cases}$$

Remark 4.10. The normalized singly-resonant system has an $n-1$ torus-symmetry $\mathbb{T}^{n-1} \ni (\phi_1, \phi_3, \ldots, \phi_n)$, with action

$$\zeta : (\phi, z) \mapsto (\mathrm{e}^{2\pi \mathrm{i} q \phi_1} z_1, \mathrm{e}^{2\pi \mathrm{i} p \phi_1} z_2, \mathrm{e}^{2\pi \mathrm{i} \phi_3} z_3, \ldots, \mathrm{e}^{2\pi \mathrm{i} \phi_n} z_n).$$

4.3.2 Second normalization

In the proof of Theorem 4.2, the functions F_m were unique up to terms in $\ker \mathrm{ad}_{H_2}$. Part of this non-uniqueness reflects the fact that Hamiltonians have infinitely many invariants under symplectic transformations.[4] However, different choices of F_m *do* change the final normal form, since a term in $\ker \mathrm{ad}_{H_2}$ may still give a contribution to the sum (4.4) via terms of higher order than H_2 in H. Pursuing this idea leads to *unique normal forms*. The result for systems in $1:2$ resonance can be found in [SvdM92]. In Chap. 3 the following result, which is a consequence of the unique $1:2$-resonant normal form, is used. We use the following abbreviations for the fundamental invariants:

$$\rho_1 = z_1\bar{z}_1, \quad \rho_2 = z_2\bar{z}_2, \quad \psi = \frac{1}{2}(z_1^p\bar{z}_2^{|q|} + \bar{z}_1 z_2^{|q|}), \quad \chi = \frac{1}{2\mathrm{i}}(z_1^p\bar{z}_2^{|q|} - \bar{z}_1^p z_2^{|q|}).$$

Proposition 4.11. *Suppose $\omega = p:q = 1:2$ or $1:-2$, and suppose H is a Hamiltonian with 2 degrees of freedom with quadratic part*

$$H_2 = i\rho_1 + i\omega\rho_2.$$

Let ϕ_1 be the normalizing transformation of Proposition 4.9, and suppose that the coefficients of ψ and χ in $H \circ \phi_1$ do not both vanish. Then there exists a symplectic transformation ϕ_2 such that

$$H \circ \phi_2 = H_2 + f(\rho_1, \rho_2, \psi).$$

Proof: (Sketch) By Proposition 4.9 there exists a ϕ_1 that brings H in the form

$$H_2 + f_0(\rho_1, \rho_2, \psi, \chi). \tag{4.6}$$

The second normalization is done by applying successive coordinate transformations $X^1_{F_s}$ for increasing s, similar to the ordinary Birkhoff procedure. The homogeneous degree-s generator F_s are now required to lie in $\ker \mathrm{ad}_{H_2}$. Jacobi's identity says

$$\{H_2, \{F, G\}\} + \{F, \{G, H_2\}\} + \{G, \{H_2, F\}\} = 0,$$

and it follows that $\ker \mathrm{ad}_{H_2}$ is closed under the operation of taking Poisson-brackets. Hence, by (4.4), transformations generated by the F_s do not disturb the general shape (4.6) of the normal form. To see how these transformations act on $H \circ \phi_1$, we look at the first nonzero homogeneous part with degree higher than H_2, which is $a\psi + b\chi$. First apply a linear symplectic transformation $z_1 \mapsto \zeta z_1, z_2 \mapsto z_2$ with $|\zeta| = 1$ to reduce to the case $b = 0$.

[4] For example, let $H(p, q)$ be a 1 d.o.f. system with elliptic equilibrium at the origin such that $H(0, 0) = 0$, and let $A(h)$ be the surface area bounded by the level curve $H = h$, then the Taylor coefficients of $A(h)$ around some $h > 0$ are (*symplectic*, not *dynamic*) invariants of H.

Since $\psi^2+\chi^2-\rho_1\rho_2^2=0$, it is possible to write $f_0(\rho_1,\rho_2,\psi,\chi)=f_0'(\rho_1,\rho_2,\psi)+\chi f_0''(\rho_1,\rho_2,\psi)$. It therefore suffices to remove all terms of the form $\rho_1^t\rho_2^u\psi^v\chi$. Taking the Poisson-bracket of the monomials $\rho_1^n\rho_2^m\psi^k$ and $2i\psi$ yields

$$\begin{aligned}\{2\mathrm{i}\psi,\rho_1^n\rho_2^m\psi^k\}&=2\rho_1^{n-1}\rho_2^{m-1}(2m\rho_1-n\rho_2)\chi\psi^k,\\ \{2\mathrm{i}\psi,\rho_1^n\psi^k\}&=-2n\rho_1^{n-1}\chi\psi^k,\\ \{2\mathrm{i}\psi,\rho_2^m\psi^k\}&=4m\rho_2^{m-1}\chi\psi^k.\end{aligned}$$

This shows that indeed all relevant terms can be removed, provided that the coefficients of ψ and χ in $H\circ\phi_1$ do not vanish simultaneously. ∎

4.4 Algorithms for the Birkhoff normal form

An important part of the Birkhoff normal form computation, is the transformation of the Hamiltonian by a symplectic coordinate change. Related methods for vector fields, on which the Hamiltonian versions are based, can be found in e.g. [Tak74c, Bro81]. The oldest methods for transforming coordinates of Hamiltonian systems is the mixed-variable generating function method, also called the Hamilton-Jacobi method or Poincaré-von Zeipel method; see [Arn89, Car81, LL92, How77, RA87, SV85]. This method seems however less suited to computer implementation (see [LL92]).

Today, the most widely-used method for computing Birkhoff normal forms is an algorithm by Deprit [Dep69, DHPR69], known also as the *Lie-triangle algorithm*. Recent expositions are [MH92, LL92, GG78]. Several implementations are available, e.g. see [ASJ93] or [RA87]. Various variations of Deprit's algorithm are known in literature [CR89, Hen70], one of which is particularly fast on Hamiltonians of special form [Hen73].

A third class of methods is based on the exponential formula (4.4), and are (distantly) related to Kolmogorov's quadratically convergent method. These methods have a lower time-complexity than the Lie-triangle algorithms. In [How77, HR84, DF76, Car81], essentially this method is presented, though not in ways that suggest efficient implementations on a computer. In this section we give Deprit's algorithms, and an algorithm based on (4.4), and discuss their computational complexity.

Analyzing the complexity of the algorithms is not straightforward. The algorithms treated below use two main types of algebraic manipulations: calculation of Poisson brackets, and solving the adjoint 'equation' $\mathrm{ad}_{H_2}F\in\mathcal{G}$. The latter is the same for all algorithms, so the number of Poisson-bracket calculations is a better indicator of the algorithm's efficiency. However, the time required for doing one such calculation depends on the number of terms in the factors: The number of terms in the bracket equals, roughly, the product of those in the factors. As a compromise, we chose to count the number of Poisson-bracket calculations performed on *homogeneous* factors. In fact this is measure of complexity is commonly used; see e.g. [Car81, BP97, Hen70].

4.4.1 Another formulation of Birkhoff's result

Usually Birkhoff's normal form theorem is formulated as a *perturbation* result, where the Hamiltonian to be normalized depends on a small parameter ϵ, and for $\epsilon = 0$ the Hamiltonian is integrable. The following result is similar to Theorem 4.2. The 'interface' is provided by setting $\mathcal{G} = \mathcal{G}_3 \oplus \mathcal{G}_4 \oplus \cdots$ and having H_i be homogeneous of degree $i+2$, then also $\widetilde{H}_i$ is homogeneous of degree $i+2$, and putting $\epsilon = 1$ in the end is allowed. This procedure is known as 'scaling', and since we use formal power series convergence in ϵ is not an issue.

Theorem 4.12. *Let $H = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \cdots$ be a Hamiltonian, with $H_i \in \mathcal{H}$, and such that H_0 has no linear part. Let $\mathcal{G} \subseteq \mathcal{H}$ be such that $\operatorname{Im} \operatorname{ad}_{H_0} \oplus \mathcal{G} = \mathcal{H}$. Then there exists a formal (in ϵ) symplectic coordinate transformation ϕ such that*

$$H \circ \phi = H_0 + \epsilon \widetilde{H}_1 + \epsilon^2 \widetilde{H}_2 + \cdots$$

with $\widetilde{H}_i \in \mathcal{G}$ for $i = 1, 2, \ldots$.

Proof: The proof is analogous to that of Theorem 4.2. ∎

4.4.2 Deprit's algorithm

Two versions of Deprit's algorithm are around. Both have as output a generating function $W(\epsilon) = W_0 + \epsilon W_1 + \epsilon^2 W_2 + \cdots$. Here ϵ is interpreted as a time variable, and the W_i are chosen such that $H \circ X^{\epsilon}_{W(\epsilon)} = \widetilde{H}$, where $X^{\epsilon}_{W(\epsilon)}$ is the time-ϵ flow of the 'time'-dependent phase flow of the Hamiltonian W. The central part of both algorithms is a recipe for computing the composition $H \circ X^{\epsilon}_{W(\epsilon)}$.

Slow version The first version of Deprit's algorithm can be summarized as follows. For the non-autonomous version see e.g. [RA87, LL92]; here we stick to the autonomous case.

$$W = \sum_{i=1}^{\infty} \epsilon^{i-1} W_i, \qquad \widetilde{H} = \sum_{i=0}^{\infty} \epsilon^i K_i,$$

$$S_0 = \text{identity operator}, \qquad S_n = \frac{1}{n} \sum_{i=0}^{n-1} \operatorname{ad}_{W_{n-i}} S_i \qquad (n \geq 1),$$

$$K_n = H_n + \frac{1}{n} \sum_{i=0}^{n-1} \left(\operatorname{ad}_{W_{n-i}} K_i + i S_{n-i} H_i \right). \tag{4.7}$$

This algorithm is derived by formally solving the differential equation (in ϵ) for the operator 'composition with $X^{\epsilon}_{W(\epsilon)}$', resulting in the expressions for S_n, and calculating what new Hamiltonian corresponds to H in the new variables. See [Car81] for a derivation.

The S_n are operators. On a computer these are implemented as recursive subroutines. Computing S_n requires $2^n - 1$ Poisson-brackets, computing K_n involves $n + \sum_{i=1}^{n-1}(2^{n-i} - 1) = 2^n - 1$ brackets, and computing $\widetilde{H}$ up to $O(\epsilon^{n+1})$ terms takes $2^{n+1} - (n+2)$ brackets. For large n then, this algorithm is slow.

Fast version By merging the differential equation for the S_n with the formulas for the Hamiltonian in the new coordinates, Deprit arrives at a faster algorithm with the same functionality. (See [Dep69], or [MH92] for a clear derivation.) It uses a triangular array of intermediate Hamiltonians H_j^i, and is summarized as follows:

$$W = \sum_{i=0}^{\infty} \epsilon^i W_i, \qquad H = \sum_{i=0}^{\infty} \epsilon^i H_i^0, \qquad \widetilde{H} = \sum_{i=0}^{\infty} \epsilon^i H_0^i,$$

$$H_j^i = \frac{1}{i}\left((j+1)H_{j+1}^{i-1} + \sum_{k=0}^{j} \mathrm{ad}_{W_k} H_{j-k}^{i-1}\right). \tag{4.8}$$

Proposition 4.13. *Algorithm (4.8) performs* $\frac{1}{6}n(n+1)(n+2)$ *Poisson-bracket computations to compute* $\widetilde{H}$ *up to* $O(\epsilon^{n+1})$ *terms.*

Proof: Computing H_j^i costs $j+1$ Poisson-brackets, and requires H_k^{i-1} to be known for $k = 0, \ldots, j+1$. To compute H_0^i for $i = 0, \ldots, n$ we therefore need the upper-triangular part of the matrix H_j^i with $i + j \leq n$, computed at a cost of $\sum_{i=1}^{n}\sum_{j=0}^{n-i}(j+1) = \frac{1}{6}n(n+1)(n+2)$ Poisson-brackets. ∎

Remark 4.14. *(Efficiency)* It is not necessary to store the entire array H_j^i in memory. By traversing it suitably, it suffices to store two rows.

Remark 4.15. *(Homogeneous perturbations)* In [Hen73], Henrard develops a modification of algorithm (4.8). This algorithm has the same asymptotic order as (4.8), however the actual number of Poisson brackets is often lower in practice, and if the perturbation is *homogeneous*, only n^2 Poisson bracket computations are necessary. For such Hamiltonians, Henrard's algorithm also outperforms the algorithm of Sect. 4.4.3.

Remark 4.16. *(Solving the adjoint equation)* Above algorithms do not constitute methods for computing the normal form yet. What fails is a method to construct W. In practice this is an additional calculation that can be done while computing the composition, see e.g. [Lun94]. If diagonalizing complex coordinates are used, this computation is straightforward.

4.4.3 The exponential-map algorithm

The idea of this algorithm is to write the normalizing transformation as a composition of transformations, each normalizing the Hamiltonian at a specific order of ϵ. These transformations are the flows of generating Hamiltonians $\epsilon^k W_k$, as in Deprit's case. The final transformation therefore is of the form

$$\phi = \epsilon X^1_{W_1} \circ \epsilon^2 X^1_{W_2} \circ \cdots$$

This should be contrasted to the form of the transformation in Deprit's case,

$$\phi = X^\epsilon_{W_1+\epsilon W_2+\epsilon^2 W_3+\cdots}$$

It turns out that the former coordinate transformation (composed with a Hamiltonian function H) can be computed more efficiently than the latter. Our algorithm is based on the proof of the normal form theorem for vector fields in [Tak74c]. It resembles Dragt and Finn's algorithm [DF76], with the difference that we evaluate all operators directly instead of postponing evaluation to the end. This results in a dramatic increase of efficiency. In fact Dragt and Finn's algorithm is not very efficient, asymptotically: in [Car81] the estimate $\exp(\pi\sqrt{2n/3})/(4n\sqrt{3})$ is given for the number of Poisson brackets to compute to order n, comparing favorably with (4.7) but not with (4.8).

Howland [How77, HR84] calculates normal forms using a method related to Kolmogorov's quadratically convergent procedure. Quadratic convergence translates to $O(k^2 \log k)$ Poisson-bracket computations, which makes it of the same order as the algorithm proposed in this section. Howland does not show how his method could be implemented, however.

The exponential-map algorithm is based on the proof of Theorem 4.2, and formula (4.4); see algorithm 4.17. Note that $K_0 = H_0$ throughout the algorithm, and is referred to only in the first line of the inner for-loop when $j = 0$: $-\operatorname{ad}_{K_j}(F_i) = -\operatorname{ad}_{H_0}(F_i) = \operatorname{ad}_{F_i}(H_0)$. This value can be computed easily while computing F_i. The algorithm can therefore be slightly rewritten so that this ad-computation is not needed, which also removes the need of initializing K_0, but for clarity this has not been done. Note also that the K_i is the only storage needed, and memory requirements are less than for Deprit's algorithm by a factor 2.

Algorithm 4.17. *(Birkhoff normal form)*
Input: $H_i \in \mathcal{H}$ *for* $i = 0, \ldots, k$, *subspace* $\mathcal{G} \subset \mathcal{H}$ *s.t.* $\operatorname{Im}\operatorname{ad}_{H_0} \oplus \mathcal{G} = \mathcal{H}$
Output: $F_i, \widetilde{H}_i \in \mathcal{H}$ *for* $i = 1, \ldots, k$, *such that*

$$\begin{aligned} &\widetilde{H}_i \in \mathcal{G} \qquad (i = 1, \ldots, k) \\ &(H_0 + \epsilon H_1 + \cdots + \epsilon^k H_k) \circ X^1_{\epsilon F_1} \circ \cdots \circ X^1_{\epsilon^k F_k} = \\ &\qquad = H_0 + \epsilon \widetilde{H}_1 + \cdots + \epsilon^k \widetilde{H}_k + O(e^{k+1}) \end{aligned}$$

Complexity: $O(k^2 \log k)$ *Poisson-brackets are computed.*

Algorithm:

$K_i \leftarrow H_i \qquad (i = 0, \dots, k)$
For i *from* 1 *to* k, *do the following:*
  Find F_i *such that* $\mathrm{ad}_{H_0}(F_i) - K_i \in \mathcal{G}$
  For j *from* $k - i$ *down to* 0, *do the following:*
    $F \leftarrow -\mathrm{ad}_{K_j}(F_i)$
    $K_{i+j} \leftarrow K_{i+j} + F$
    $q \leftarrow 2$
    While $qi + j \le k$ *do the following:*
      $F \leftarrow \mathrm{ad}_{F_i}(F)/q$
      $K_{qi+j} \leftarrow K_{qi+j} + F$
      $q \leftarrow q + 1$
    EndWhile
  EndFor
EndFor
Output F_i, *and* $\widetilde{H}_i = K_i$.

Proof of the algorithm: We prove correctness by proving invariance, over the outer for-loop, of the following assertion:

(4.9)
$$\begin{cases} (H_0 + \epsilon H_1 + \cdots + \epsilon^k H_k) \circ X^1_{\epsilon F_1} \circ \cdots \circ X^1_{\epsilon^i F_i} = \\ \qquad\qquad K_0 + \epsilon K_1 + \cdots + \epsilon^k K_k + O(\epsilon^{k+1}), \\ K_j \in \mathcal{G}. \qquad (j = 1, \dots, i) \end{cases}$$

After initialization and for $i = 0$, (4.9) is trivially true. For $i = k$ it implies the output condition on F_i and $\widetilde{H}_i$. So we are done if we prove invariance of (4.9).

The inner for-loop computes $(K_0 + \epsilon K_1 + \cdots + \epsilon^k K_k) \circ X^1_{\epsilon^i F_i}$. By (4.4), and using linearity of the ad-operator, the order-ϵ^i term in this expression is $K_i + \mathrm{ad}_{F_i}(H_0) = K_i - \mathrm{ad}_{H_0}(F_i)$. By the choice of F_i, indeed $K_i \in \mathcal{G}$ after this pass through the outer for-loop, and K_i does not change in subsequent passes.

Each pass through the inner for-loop computes $\epsilon^j K_j \circ X^1_{\epsilon^i F_i}$ for some j, up to $O(\epsilon^{k+1})$ terms. By (4.4), this is a sum of terms of the form $\epsilon^{qi+j} \mathrm{ad}^q_{F_i}(K_j)/q!$ for $q = 0, \dots, \infty$, and this is just what is computed recursively in the inner While-loop, for q-values that contribute to terms of order k or less. By having the counter j counting downwards, the high order K_j's are computed first, eliminating the need of storing the 'new' K_j's in a separate array to prevent overwriting the old ones. This proves invariance of (4.9), and correctness of the algorithm.

The number of Poisson-brackets computed by the algorithm can be estimated as follows. Through one pass of the inner for-loop, at most $(k-j)/i$ brackets are computed. Summing this over $j = 0, \dots, k-i$ this becomes $k(k+1)/(2i) - (i-1)/2$ for the number of brackets computed in the inner for-loop. The total number of brackets is then estimated by

$$\sum_{i=1}^{k} \frac{k(k+1)}{2i} - \frac{i-1}{2} \leq \frac{k(k+1)}{2}\mathrm{Harm}_k - \frac{k(k-1)}{4} = O(k^2 \log k),$$

where Harm_k is the kth harmonic number, proving the asymptotic order of the algorithm. ∎

Remark 4.18. *(Normalizing transformation)* As it stands, the algorithm only computes the normal form, not the normalizing coordinate transformation. This transformation is obtained by evaluating $I \circ X^1_{\epsilon^3 F_3} \circ \cdots \circ X^1_{\epsilon^n F_n}$, where I is the identity transformation, which is very similar to the computation of $H \circ X^1_{\epsilon^3 F_3} \circ \cdots$ done by the algorithm. Slight modifications are necessary to reflect the fact that I has linear terms, whereas H is quadratic to leading order. The resulting algorithm takes approximately twice as much time as algorithm 4.17, hence is also $O(n^2 \log n)$.

5 Singularity theory

The space of functions – or maps – is huge. Fortunately, many of its elements may be regarded as equivalent in a natural way, for example under right- or left-right transformations. Singularity theory studies how such equivalences foliate these spaces into a more manageable family of orbits of equivalent functions or maps.

5.1 Overview

Singularity theory plays a pivotal role in this work. It provides the theorems to prove existence of normalizing transformations to the one-degree-of-freedom model systems, which is the subject of Chaps. 2 and 3. In Chap. 7 the constructive counterparts of these theorems are used to compute the normalizing transformations. For this to work, the bases for the tangent spaces involved must be brought into so-called *standard* form. The algebraic structure of these spaces depends on the equivalence class (e.g. right-transformations, left-right transformations), and motivated the search for the appropriate generalizations of Gröbner bases in Chap. 6.

This chapter collects the results on singularity theory that are used throughout. Most of the results were taken from the rather technical [Mar82, Mat68]. More accessible introductions are [BL75, BG84, GG73, Gib79, Lu76, Was74], and [GS79, PS78, Sma67, Tho72] give much motivation and philosophy. Though we do not use it here, classification of singularities touch at the core of the subject; see e.g. [Arn81, Arn93b, Sie74, Sie73]. The relation of singularity theory and symmetries has also been extensively studied; see e.g. [BF91, GS85, GSS88, Poè76, BHvNV99]

We note here that the Mather-Malgrange preparation theorem is only referred to in passing. This important theorem bridges the gap between finite-dimensional deformation theory, summarized in Sect. 5.2, and deformation theory in the smooth category. This chapter indeed deals with C^∞ functions and maps, but our interest lies in the computation of finite parts of reparametrizations. For this purpose, it makes sense to work with (truncated) formal power series, and Chaps. 6 and 7 are set in this context. Regarding the relevance of the current chapter for those, via the jet-map and Borel's theorem, it is an exercise to show that the main results of singularity theory for smooth functions and maps, immediately

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imply those for the formal setting. In fact the formal results may be proven directly without much difficulty, and indeed the results of Chaps. 6 and 7 can be viewed as constructive (i.e., algorithmic) proof of the formal versions of theorems 5.5 and 5.22.

The setup of this chapter is as follows. The first section is an overview of the finite-dimensional analogue of singularity theory, giving a palatable introduction of the fundamental concepts. The next section presents singularity theory of functions, with right-transformations as the equivalence relation. This section also deals with BCKV normal forms, a specific kind of deformations geared to the application of Chap. 2. Finally, the elements of singularity theory dealing with maps under left-right transformations, necessary for analyzing the energy–momentum map in Chap. 3, are presented in Sect. 5.4.

5.2 Introduction: The finite dimensional case

Almost every concept used in the singularity theory of functions and maps, has a direct counterpart in the finite dimensional context of Lie groups acting on smooth finite-dimensional manifolds. In this finite-dimensional setting, the proofs are straightforward and only involve the implicit function theorem. Since this theory is so technically undemanding but nonetheless conceptually rich, it is a good introduction to sections 5.3 and 5.4. Our main source for this section was [Gib79].

Let M be a manifold, G a Lie group, and suppose that both are smooth, finite dimensional manifolds. Let $\zeta : G \times M \to M$ be a smooth action of G on M. Instead of $\zeta(\gamma, f)$ we sometimes simply write γf. For a given point $f \in M$, the action ζ gives rise to an orbit, in this notation given by Gf. We are interested in the tangent space to this orbit at the point f, which we denote by $T_f(Gf)$.

Let $\zeta_f : G \to M$ denote the map $\gamma \mapsto \zeta(\gamma, f)$. Its image is the orbit Gf. The tangent space to Gf at $f = \zeta_f(Id)$ is just the image under the differential $D\zeta_f$ of the tangent space $T_{Id}(G)$ to G at the identity element:

$$T_f(Gf) = D_{Id}\zeta_f \left(T_{Id}(G)\right).$$

(Here, and elsewhere in this chapter, D_x denotes the (total) differential at x, not the differential with respect to x.) The codimension of $T_f(Gf)$ in $T_f(M)$ is also called the *codimension of* f. If this codimension is 0 the inverse function theorem can be applied, with the result that for every f' in some neighborhood of f, there exist $\gamma \in G$ such that $f' = \zeta(\gamma, f) = \gamma f$. We say that f' is *equivalent* to f, and write $f' \sim f$. Concisely,

Definition 5.1. $f' \sim f \quad \Leftrightarrow \quad \exists \gamma \in G : f' = \gamma f$

An element f with the property that it is equivalent to every f' in its neighborhood is called a *stable* element. Note that the dimension of $T_f(Gf)$ is never larger than the dimension of G itself, therefore G has to be 'large' enough in order for stable elements to exist.

5.2.1 Deformations

Stable elements, with codimension 0, form the simplest case. Now let us proceed, and suppose that the codimension of f is nonzero, say equal to d. Small changes to f along orbits of G will *not* change f's equivalence class; however changes *transversal* to its G-orbit will. A catalog of representatives of all equivalence classes that occur in a neighborhood of f is given by a transversal section of the orbit Gf at f. Such a transversal section, which is a submanifold in M of dimension d, can be parametrized as a d-parameter family of elements in M. Families depending on parameters are called *deformations*:

Definition 5.2. *A* d-parameter deformation *of* f *in* M *is a map* $F(u) : \mathbb{R}^d \to M$ *such that* $F(0) = f$,

and the infinitesimal directions in which $F(0) = f$ is deformed are called "deformation directions":

Definition 5.3. *Let* $F(u) : \mathbb{R}^d \to M$ *be a deformation of* $F(0) = f$. *The elements* $\frac{\partial F}{\partial u_i}|_{u=0} \in T_f(Gf)$ *are called the* deformation directions *of* F.

Note that this definition is coordinate-dependent.

Instead of *deformation direction* the synonym *initial speed* is also used; and instead of *deformations* the term *unfoldings* is used often. The latter are sometimes defined a little differently, see e.g. [Mar82], the difference being mainly notational, see e.g. [Mon94].

Now, the deformation F is a transversal section if its deformation directions complement the tangent space, or symbolically:

$$T_f(Gf) \oplus D_0F(\mathbb{R}^d) = T_f(M). \tag{5.1}$$

Deformations for which (5.1) hold are called *transversal deformations*. Now let $G(v) : \mathbb{R}^k \to M$ be some deformation of f. It is is said to be *induced* from F if there exists a reparametrization $h : \mathbb{R}^k \to \mathbb{R}^d$ with $h(0) = 0$, and a deformation $I : \mathbb{R}^k \to G$ of the identity element in G, such that

$$G(v) = I(v)F(h(v)).$$

If it happens that *every* deformation G of f can be induced from F in this way, then F is called a *versal* deformation. Again by the inverse function theorem it can be shown that a deformation is versal if it is transversal, see e.g. [Gib79, p. 90]. The "only if" direction is trivial. Sometimes the notion of *universal deformation* is used; this is a versal deformation with a minimal number of parameters.

It thus turns out that the infinitesimal data of the tangent space T_f is enough to write down a versal deformation, which captures all possible behavior of the singularity f in a full neighborhood of it, modulo the equivalences of the Lie group.

5.3 Functions and right-transformations

We now turn to the case of smooth functions on $\mathbb{R}^n$ acted upon by the group of right-transformations. Neither the manifold of smooth functions nor the group is finite-dimensional, and the previous section does not apply as it stands. It is a remarkable fact that the results do continue to hold true for this case. The proofs are much more difficult, however, since the inverse function theorem cannot be used; see e.g. [Mar82, BL75] for proofs and more details. Additional complications arise from the necessity of truncating when doing actual computations. Lots of results exist that guarantee sufficiency of certain truncation-orders in several special cases. These can be regarded as generalizations or extensions of the Morse lemma, which states[1] that if a germ has a nondegenerate quadratic part, it is equivalent to the germ obtained by truncating at degree 2. Of this class of results we apply only two, namely propositions 5.7 and 5.8.

In contrast to context of smooth functions, for formal power series the proof of the main theorem (theorem 5.5, the equivalence of transversal and versal deformations) is relatively straightforward. In fact the results of Chap. 7 can be regarded as constructive proof. (See also remark 5.23 below.) Another straightforward way of proving the results in the formal context is to proceed directly from the smooth results, using the jet-map and Borel's theorem. We shall not make this explicit, however, and be satisfied with the algorithms of Chap. 7.

Let us introduce the main players in more detail. The manifold M is the set of Γ-invariant C^∞ functions on $\mathbb{R}^n$, symbolically $M = \mathcal{E}_n^\Gamma$. Here Γ is some compact (usually finite) Lie group with a linear action on $\mathbb{R}^n$. The group of transformations G that act on M is the group of origin preserving Γ-equivariant C^∞ maps on $\mathbb{R}^n$, and they act on elements of M by composition on the right. The results of this section are well-known for the case that $\Gamma = \{Id\}$, and straightforward generalizations otherwise; see [Was75, BHLV98].

5.3.1 Equivalence and versal deformations

Definition 5.4. *$\mathcal{E}_n^\Gamma$ is the set of germs of smooth Γ-invariant functions on $\mathbb{R}^n$.*

$\mathbf{V}_n^\Gamma$ is the $\mathcal{E}_n^\Gamma$-module of germs of smooth Γ-equivariant vector fields vanishing at the origin.

The tangent space to the group of right-transformations G at the identity mapping is (isomorphic to) the module of vector fields $\mathbf{V}_n^\Gamma$. Now the tangent space $T_f(Gf)$ to the orbit of f, again abbreviated to T_f, is:

$$T_f := T_f(Gf) = D_{Id}\zeta_f(T_{Id}(G)) = \{Xf | X \in \mathbf{V}_n^\Gamma\}.$$

The notion of *deformation* of a d-parameter function (or, more precisely, of a *germ* of a function[2]) f is the straightforward analogue of a deformation in the

[1] Actually it implies a little more, see e.g. [GG73, Thm. 6.9]

[2] Note the difference between germs of deformations, and deformations of germs: Transformations between the former may shift the origin as a function of the parameters. We use deformations of germs, requiring transformations to keep the origin fixed.

finite-dimensional context: a map from $\mathbb{R}^d$ to the space of functions $M = \mathcal{E}_n^\Gamma$. Again, F is called a *transversal deformation* if

$$T_f + D_0F(\mathbb{R}^d) = T_f(M).$$

In particular, f has a transversal deformation if and only if $T_f \subseteq T_f(M)$ has finite codimension. Secondly, a deformation $F(u) : \mathbb{R}^d \to M$ is called a *versal deformation* of $f = F(0)$ if for any deformation $G(v) : \mathbb{R}^q \to M$ there exist maps ϕ and h, with

$$\phi : \mathbb{R}^n \oplus \mathbb{R}^q \to \mathbb{R}^n, \qquad h : \mathbb{R}^q \to \mathbb{R}^d,$$

such that $G(x, v) = F(\phi(x, v), h(v))$. It is easy to show that a versal deformation is transversal. The converse is much more difficult to prove:

Theorem 5.5. *Suppose $f \in M$ is a germ of a Γ-invariant C^∞ function, and $F : \mathbb{R}^d \to M$ is a transversal deformation of f. Then F is a versal deformation of f*

For a proof, see [Was74, Thm 3.19]. Note that existence of a transversal deformation implies that f is finitely determined, by [Was74, Thm. 2.6] or [Mat68, Thm. 3.5]. For more information see also [Sie74].

We conclude this section by giving a few effective conditions for equivalence of functions. The adjective 'effective' refers to the fact that these conditions can be checked by doing calculations in finite-dimensional vector spaces. The propositions below are used in Sect. 5.3.2.

Definition 5.6. *a) j^k is the jet map of order k, mapping germs in $\mathcal{E}_n^\Gamma$ to their Taylor polynomial up to and including order k.*
b) $\mathfrak{m}_k := \{f \in \mathcal{E}_n^\Gamma | j^{k-1}(f) = 0\}$

In particular $j^0(f)$ is the constant part of f, and $\mathfrak{m}_1 = \mathfrak{m}$, the maximal ideal in $\mathcal{E}_n^\Gamma$. If $\Gamma = \{Id\}$ then $\mathfrak{m}_k = \mathfrak{m}^k$. We stick to the case that $\Gamma = \{Id\}$ a little longer, and drop the Γ from the notation. Note that $\mathbf{V}_n$ is generated, as an $\mathcal{E}_n$-module, by $x_i \frac{\partial}{\partial x_j}$ with $1 \leq i, j \leq n$. This means that the tangent space T_f is just $\mathfrak{m} \cdot J(f)$, where $J(f)$ is the Jacobian ideal of f:

$$J(f) := \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle_{\mathcal{E}_n}.$$

The following proposition gives conditions under which (non-symmetric) germs are equivalent:

Proposition 5.7. *[Mar82, III.4.2] Let $f, g \in \mathcal{E}_n$, and assume that $g - f \in \mathfrak{m}_k$, i.e., $j^{k-1}(g - f) = 0$.*

a) If $T_f \supset \mathfrak{m}_k$ then $g \sim f$ provided that $j^k(g-f)$ is small enough.
b) If $\mathfrak{m} \cdot T_f \supset \mathfrak{m}_k$ then $g \sim f$.

(Here '$j^k(g-f)$ small enough' means that the coefficients of the kth order Taylor polynomial of $g-f$ at the origin are sufficiently small.) The analogous result for germs with symmetry is this:

Proposition 5.8. *Let $f, g \in \mathcal{E}_n^\Gamma$, and suppose that $j^{k-1}(g-f) = 0$. Let M denote the finite-dimensional vector space $\mathfrak{m}_k/(\mathfrak{m} \cdot \mathfrak{m}_k)$, and set*

$$M_m := M \cap (\mathfrak{m}_m/\mathfrak{m}_{m+1}) = \{h \in M : h \text{ is homogeneous of degree } m\}.$$

a) Suppose that $T_f \supset \mathfrak{m}_k$ then $g \sim f$ provided that the projection of $g-f$ into M is sufficiently small.
b) Suppose that $\mathfrak{m} \cdot T_f \supset \mathfrak{m}_k$ then $g \sim f$.
c) Suppose that $T_f \supset \mathfrak{m}_k$. Suppose further that the projection of f into M is an element of M_k. Then $g \sim f$ provided that $j^k(g-f)$ is sufficiently small.

See appendix A.2 for the proof.

5.3.2 Applications

The following examples are used in Chaps. 2 and 3, and use the propositions of the previous section.

The Γ-invariant Morse lemma As a first application of proposition 5.8, we extend the Morse lemma (see e.g. [Mar82, GG73]) to the Γ-invariant case.

Proposition 5.9. *Let $f \in \mathcal{E}_n^\Gamma$ be a germ without linear part, and a nondegenerate quadratic part. Then, for any $g \in \mathcal{E}_n^\Gamma$ with $j^2(g-f)$ small enough, $f \sim g$.*

Proof: Let f_2 be the quadratic part of f. Since f_2 is nondegenerate, we have $T_{f_2} = \mathfrak{m}_2$. As $j^2(f-f_2) = 0$ by definition, proposition 5.8 (c) implies that $f \sim f_2$. Since $j^2(g-f_2) = j^2(g-f)$ is small, applying the same proposition again yields $g \sim f_2$. Together this implies $f \sim g$. ∎

Remark 5.10. This result also follows from the symmetric splitting lemma, which is given in e.g. [BF91, App. A]

The hyperbolic umbilic The next example is used in the $1:2$ planar reduction, to produce a normal form of a singular $\mathbb{Z}_2$-symmetric germ with vanishing quadratic, but non-vanishing third order part. This result is used in section 2.3.2 in the proof of proposition 2.12.

Proposition 5.11. *Let $\mathbb{Z}_2$ be a group with $\mathbb{R}^2$-action $(x,y) \mapsto (x,-y)$. Let f be the $\mathbb{Z}_2$-symmetric germ $f = x(\alpha x^2 + \beta y^2) + h.o.t.$, and assume $\alpha \neq 0$, $\beta \neq 0$. Then f is isomorphic to $\pm x(x^2+y^2)$.*

Proof: By a linear transformation we may assume that α and β are in fact both $+1$ or -1. The tangent space T_g with $g = \pm x(x^2+y^2)$ is generated by $x\frac{\partial g}{\partial x} = 3x^3 + xy^2$, $y^2\frac{\partial g}{\partial x} = 3x^2y^2 + y^4$ and $y\frac{\partial g}{\partial y} = 2xy^2$. Equivalently, $T_g = \left\langle x^3, xy^2, y^4 \right\rangle_{\mathcal{E}_2^{\mathbb{Z}_2}} = \mathfrak{m}_3$. Now apply proposition 5.8(c) with $k = 3$ to g. Since g is homogeneous, $g \in M_3$, so we conclude that $f = g + h.o.t. \sim g$ for arbitrary higher order terms. ∎

Symmetric deformation of the hyperbolic umbilic We conclude this section by giving a versal deformation of the normal form $x(x^2+y^2)$ found in proposition 5.11 above. See also Sect. 7.2.4.

Proposition 5.12. *Let $\mathbb{Z}_2$ be a group acting on $\mathbb{R}^2$ by $(x,y) \mapsto (x,-y)$. Then $F(x,y;u_0,u_1,u_2) := x(x^2+y^2) + u_0 + u_1x + u_2y^2$ is a versal deformation of $x(x^2+y^2)$ in the space of $\mathbb{Z}_2$-invariant functions.*

Proof: The module of $\mathbb{Z}_2$-equivariant vector fields is generated by $\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial y}$, so that $T_f = \left\langle 3x^2+y^2, 2xy^2 \right\rangle_{\mathcal{E}_2^{\mathbb{Z}_2}} = \left\langle 3x^2+y^2, 2xy^2, x^3, y^4 \right\rangle_{\mathcal{E}_2^{\mathbb{Z}_2}}$. This shows that $T_f + \mathrm{span}_{\mathbb{R}}\{1,x,y^2\} = \mathcal{E}_2^{\mathbb{Z}_2}$. By definition this means that F is a transversal deformation of $x(x^2+y^2)$. An application of theorem 5.5 completes the proof. ∎

Remark 5.13. These examples are the tip of an iceberg. For more on classification of singularities see e.g. [Arn81, BF91, BL75, BG84, GS85, GSS88, Mar82, Sie73, Sie74].

5.3.3 BCKV normal form

The deformations appearing in Sect. 5.3.1, see for example Proposition 5.12, have two kinds of variables: the ordinary 'phase space' variables like x and y, and the deformation parameters $u_1,\ldots,u_d$. The different roles played by these variables is clearly seen in the transformations (i.e., equivalence relation) between deformations: Transformations of the phase variables may depend on both kinds of variables, whereas transformations of the deformation parameters must not depend on the phase variables.

In some applications, more than two classes of variables appear in a natural way. For example, there are problems where time, space and parameters appear and should be distinguished; see [Was75]. This section deals with yet another situation, where we have two kinds of parameters, namely *ordinary* and *distinguished* ones, the dependence between these variables in transformations of deformations is according to the following table:

Variable → / May depend on ↓	phase	distinguished	ordinary
phase	yes	no	no
distinguished	yes	yes	no
ordinary	yes	yes	yes

In Chap. 2 this situation is met. There the distinguished parameters are phase variables which are no part of the (reduced) dynamical system. Since they are phase variables, the (ordinary) parameters of the model may not depend on them. Coordinate transformations that respect this structure are called BCKV transformations (see [BCKV93, BCKV95]).

Since BCKV normal form puts restrictions on the reparametrizations, it is reasonable to expect the number of (equivalence classes of) normal forms to increase. This indeed happens, even to the extent that normal forms of finite codimension cease to exist. This problem is tackled using the *path formulation* [GS85, GS79, Mon94]. In effect this amounts to fixing a number of deformation parameters in the deformation, suitable to the problem at hand, in other words fixing a *path* through the infinite dimensional space of deformation parameters; see also [BCKV93].

The normal form and transformations are supposed to be equivariant under the action of some compact symmetry group Γ.

BCKV theory – definitions and main theorem The main point was made above: reparametrizations may not depend on the distinguished parameters. Secondly, in the intended application, the distinguished parameter is an *angular momentum*, and therefore nonnegative. Hence, we also require the reparametrization of the distinguished parameter to respect the zero level. These ingredients lead to definition 5.15 below.

Remark 5.14. *(Notation)* With BCKV deformations, there are three levels of variables: *phase* variables $x \in \mathbb{R}^n$, *distinguished* parameters $\lambda \in \mathbb{R}^r$ and *ordinary* parameters $u \in \mathbb{R}^s$. We abbreviate $\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^s$ by $\mathbb{R}^{n+r+s}$, and similarly for subspaces.

Definition 5.15. *(BCKV-restricted morphisms:) Let two deformations $F \in \mathcal{E}^\Gamma_{n+r+s}$ and $G \in \mathcal{E}^\Gamma_{n+r+t}$ of $f = F(\cdot,\cdot,0) \in \mathcal{E}^\Gamma_{n+r}$ be given, such that $f(0,\lambda) = 0$. F is said to be* induced from *G by Γ-equivariant* BCKV-restricted morphisms *if there exist germs of Γ-equivariant mappings $\Psi : \mathbb{R}^{n+r+s} \to \mathbb{R}^{n+r+t}$, $\Phi : \mathbb{R}^{r+s} \to \mathbb{R}^{r+t}$ and $\Theta : \mathbb{R}^s \to \mathbb{R}^t$ such that the following diagram commutes:*

$$\begin{array}{ccccccccc}
\mathbb{R}^{n+r} \times \{0\}^s & \xrightarrow{\hookrightarrow} & \mathbb{R}^{n+r+s} & \xrightarrow{\pi_1} & \mathbb{R}^{r+s} & \xrightarrow{\pi_2} & \{0\}^r \times \mathbb{R}^s & \xrightarrow{\hookrightarrow} & \mathbb{R}^{r+s} \\
\mathrm{Id}\downarrow & & \Psi\downarrow & \searrow^{F}\ \nearrow_{G} \mathbb{R} & \Phi\downarrow & & \Theta\downarrow & & \Phi\downarrow \\
\mathbb{R}^{n+r} \times \{0\}^t & \xrightarrow{\hookrightarrow} & \mathbb{R}^{n+r+t} & \xrightarrow{\tilde\pi_1} & \mathbb{R}^{r+t} & \xrightarrow{\tilde\pi_2} & \{0\}^r \times \mathbb{R}^t & \xrightarrow{\hookrightarrow} & \mathbb{R}^{r+t}
\end{array}$$

On $\mathbb{R}^{n+r}$, $\mathbb{R}^{n+r+s}$ and $\mathbb{R}^{n+r+t}$ the action of Γ is defined by trivially extending it on $\mathbb{R}^n$.

Commutativity of the diagram amounts to: There exist $\phi : \mathbb{R}^{r+s} \to \mathbb{R}^r$, $\psi : \mathbb{R}^{n+r+s} \to \mathbb{R}^n$ such that $\Phi = (\phi, \Theta)$, $\Psi = (\psi, \phi, \Theta)$, $\psi(x,\lambda,0) = x$, $\phi(\lambda,0) = \lambda$, $\phi(0,u) = 0$, $\Theta(0) = 0$, and $F(x,\lambda,u) = G(\psi(x,\lambda,u), \phi(\lambda,u), \Theta(u))$. The ϕ, ψ and

Θ are the analogs of ϕ and ρ_i of (2.12), but obey more restrictions. Morphisms (Ψ, Φ, Θ) as above are called *BCKV-restricted morphisms.*

Again, *versal* deformations are those deformations from which every other can be induced, and a *universal* deformation is a versal deformation with the minimal possible number of parameters. In [BCKV93, thm. 11], versal deformations with respect to BCKV-restricted morphisms are characterized. Next we prove the Γ-equivariant version of this:

Theorem 5.16. *(BCKV-restricted versal deformations:) Let $f \in \mathcal{E}^{\Gamma}_{n+r}$ be a family of germs of Γ-equivariant germs depending on a distinguished parameter $\lambda \in \mathbb{R}^r$. Let $f_0 \in \mathcal{E}^{\Gamma}_n : x \mapsto f(x, 0)$ have codimension c. Then*

1. *f has a universal deformation with respect to Γ-equivariant BCKV-restricted morphisms if and only if f, considered as a deformation of f_0, is versal with respect to ordinary Γ-equivariant morphisms.*
2. *If $F(x, \lambda, u)$ is a (uni)versal deformation of f with respect to Γ-equivariant BCKV-restricted morphisms, then $F(x, 0, u)$ is a (uni)versal deformation of f_0 with respect to ordinary Γ-equivariant morphisms.*
3. *If $f(x, \lambda)$ is a universal deformation of f_0 with respect to ordinary Γ-equivariant morphisms, then $r = c$ and $F : \mathbb{R}^{n+c+c} \to \mathbb{R}$ defined by*

$$F(x, \lambda, u) = f(x, \lambda) + \sum_{j=1}^{c} u_j \frac{\partial f}{\partial \lambda_j}(x, 0)$$

is a universal deformation of f with respect to Γ-equivariant BCKV-restricted morphisms.

Proof: The proof for the non-equivariant case can be carried over to the present setting with obvious changes. See [BCKV93]. ∎

Path formulation As the number of distinguished parameters is fixed, theorem 5.16 implies that if the central singularity f_0 has a high codimension, there are no versal deformations with respect to Γ-equivariant BCKV-restricted morphisms. However, we can view the system as a *subfamily* of a versally deformed system. The normal form then includes functions that describe the submanifold, embedded in the versal system's parameter space, that the system traces out. Bifurcations of the intersection of this submanifold with the bifurcation set yields additional information. This description is usually called the path formulation, see [GS85, GS79, Mon94, BF91]. For this final reduction, we need the following:

Definition 5.17. *A* BCKV-restricted reparametrization *is a mapping (ϕ, θ) with $\phi : \mathbb{R}^{r+s} \to \mathbb{R}^r$, $\theta : \mathbb{R}^s \to \mathbb{R}^s$ such that $\phi(0, u) = 0$, $\theta(0) = 0$.*

Note that it is not required that $\phi(\lambda, 0) = \lambda$. The following lemma is a slightly stronger version of [BCKV93, lemma 7], and is used in the proof of proposition 5.19 below.

Lemma 5.18. *[BCKV93] Let $r \leq s$, let $\pi : \mathbb{R}^s \to \mathbb{R}^r$ be a projection onto some r-dimensional subspace of $\mathbb{R}^s$, and let $\tilde{h} : (\lambda, u) \in \mathbb{R}^{r+s} \to \mathbb{R}^s$ be a map (a 'normal form') such that $\tilde{h}(0,0) = 0$ and the derivatives $D_\lambda(\pi \circ \tilde{h}(\lambda, u))|_{\lambda=u=0}$ and $D_u(\pi \circ \tilde{h}(\lambda, u))|_{\lambda=u=0}$ both have rank r. Then, for any $h \in \mathcal{E}(r+s, s)$ with $h(0,0) = 0$ there exists a BCKV-restricted reparametrization $\Upsilon = (\phi, \theta)$ such that*

$$\pi(h(\lambda, u)) = \pi(\tilde{h}(\Upsilon(\lambda, u))).$$

Moreover, if also $D_\lambda \pi \circ h(\lambda, u)$ and $D_u \pi \circ h(\lambda, u)$ both have rank r (at $\lambda = u = 0$), then Υ can be chosen invertible.

Proof: As $D_u \pi \circ \tilde{h}(0, u)$ has full rank, and $\pi \circ h(0,0) = \pi \circ \tilde{h}(0,0)$, by the inverse function theorem there exists a function $\theta(u)$ with $\theta(0) = 0$ such that $\pi \circ h(0, u) = \pi \circ \tilde{h}(0, \theta(u))$. Now $D_\lambda \pi \circ \tilde{h}(\lambda, \theta(u))$ has full rank, and moreover $\pi \circ h(0, u) = \pi \circ \tilde{h}(0, \theta(u))$, for all u, so, applying the inverse function theorem again, we find a function $\phi(\lambda, u)$ with $\phi(0, u) = 0$, such that $\pi \circ h(\lambda, u) = \pi \circ \tilde{h}(\phi(\lambda, u), \theta(u))$. The last remark follows by applying the lemma with the roles of h and $\tilde{h}$ interchanged. ∎

Proposition 5.19. *Let $g(x, \lambda, u) : \mathbb{R}^{n+r+s} \to \mathbb{R}$ be a generic Γ-invariant germ, and assume that $f(x, \sigma_1, \ldots, \sigma_s)$ is a universal deformation of $g(x, 0, 0)$ using unrestricted Γ-equivariant morphisms. Then there exists a BCKV-restricted reparametrization Υ such that for the normal form*

$$F(x, \lambda, u) := f(x, \lambda_1 + u_1, \ldots, \lambda_r + u_r, \tilde{\sigma}_{r+1}(\lambda, u), \ldots, \tilde{\sigma}_s(\lambda, u)),$$

where $\tilde{\sigma}_i$, $i = r+1, \ldots, s$, are some functions, we have that g can be induced from $F \circ (\pi_x, \Upsilon)$ using BCKV-restricted Γ-equivariant morphisms. Here π_x denotes the projection $\pi_x : (x, \lambda, u) \mapsto x$.

Proof: Let $h(\lambda, u)$ be a reparametrization, and $\Phi(x, \lambda, u)$ a coordinate transformation, such that $f(\Phi(x, \lambda, u), h(\lambda, u)) = g(x, \lambda, u)$. Define $\tilde{h}_i(\lambda, u) := \lambda_i + u_i$ if $1 \leq i \leq r$ and $\tilde{h}_i(\lambda, u) := u_i$ if $r+1 \leq i \leq s$, and set $\pi(\sigma_1, \ldots, \sigma_s) = (\sigma_1, \ldots, \sigma_r)$. The lemma now applies. By genericity we may assume that the relevant derivatives have rank r, so we find an invertible BCKV-restricted reparametrization Υ such that $h_i(\lambda, u) = \tilde{h}_i(\Upsilon(\lambda, u))$ for $i = 1, \ldots, r$, which means that for

$$F(x, \lambda, u) := f(x, \lambda_1 + u_1, \ldots, \lambda_r + u_r, h_{r+1} \circ \Upsilon^{-1}(\lambda, u), \ldots, h_s \circ \Upsilon^{-1}(\lambda, u)),$$

we have $g(x, \lambda, u) = F \circ (\pi_x, \Upsilon) \circ (\Phi, \pi_\lambda, \pi_u)$, where $\pi_\lambda : (x, \lambda, u) \mapsto \lambda$ and $\pi_u : (x, \lambda, u) \mapsto u$, proving the proposition. ∎

5.4 Maps and left-right transformations

In Chap. 3, the object to normalize is the *energy–momentum map*, a parameter-dependent map from phase space to $\mathbb{R}^2$. We are interested in its fibers. Since these are smoothly deformed by left-right transformations, we use these transformations to normalize the map. For more information see Sect. 3.1, and [Dui84].

5.4.1 The tangent space

In the language of Sect. 5.2, the manifold M is $\mathcal{E}_n^\Gamma \times \mathcal{E}_n^\Gamma$, the Γ-invariant $\mathcal{C}^\infty$ maps from $\mathbb{R}^n$ to $\mathbb{R}^2$. We include a symmetry group Γ, as in the application the energy–momentum map will be invariant under the circle symmetry produced by the Birkhoff normal form. The transformation group G is

$$G := \{(A, B) \in \mathcal{E}_{n,n}^\Gamma \times \mathcal{E}_{2,2} | A, B \text{ origin-preserving and invertible}\}.$$

The first component is the set of Γ-equivariant right transformations. The second component are the left-transformations. The group elements have the following action on M:

$$\zeta : \quad G \times M \to M : \quad ((A, B), \boldsymbol{E}) \mapsto B \circ \boldsymbol{E} \circ A.$$

The group operation is $(A, B) \circ (A', B') = (A' \circ A, B \circ B')$. The tangent space to G is then (isomorphic to) $\mathbf{V}_n^\Gamma \times \mathcal{E}_{2,2}$. Again, using this we can find the tangent space to the orbit of an arbitrary map $\boldsymbol{E} \in M$ under the action of G. This tangent space is denoted by $T_{\boldsymbol{E}}$:

(5.2)

$$T_{\boldsymbol{E}} := T_{\boldsymbol{E}}(G\boldsymbol{E}) = \{(\alpha \boldsymbol{E}_1 + \beta_1(\boldsymbol{E}_1, \boldsymbol{E}_2), \alpha \boldsymbol{E}_2 + \beta_2(\boldsymbol{E}_1, \boldsymbol{E}_2)) \,|\alpha \in \mathbf{V}_n^\Gamma, \beta \in \mathcal{E}_{2,2}\}.$$

Here $\boldsymbol{E}_i$ and β_i are the components of the 2-vector-valued maps $\boldsymbol{E}$ and β.

Remark 5.20. *(Fixing the origin)* In order to use germs (at 0) as left-transformation, we have to require that element of M map the origin (in $\mathbb{R}^4$) to the origin (in $\mathbb{R}^2$). This will be assumed throughout.

Remark 5.21. *(Germs and symmetry)* The group G acts on the manifold M consisting of germs at $0 \in \mathbb{R}^2$. Therefore, the action of the group must keep the origin fixed. However, when considering *germs* of *deformations* of maps, this is only required when the deformation parameters are 0, and the codimension of the tangent space 5.2 only gives an upper bound of the codimension of elements of M as deformations. But, if the symmetry group Γ is such that the origin is the only fixed point of the action of Γ, equivariance of the action of G implies that it fixes the origin, and the codimension of 5.2 is equal to the deformation-codimension.

From Mather's results [Mat68] we can deduce the following theorem. (In his notation we have $T_{\boldsymbol{E}} = (t\boldsymbol{E})(B) + (\omega \boldsymbol{E})(A)$, and $T(M) = \theta(\boldsymbol{E})$. Mather denotes the group G of left-right transformations by $\mathcal{A}$.)

Theorem 5.22. *Let $T_{\boldsymbol{E}} \subseteq T(M)$ have finite codimension d, and let $T(M)/T_{\boldsymbol{E}}$ be spanned by $t_1(x) \ldots, t_d(x)$ as a real vector space. Then*

$$\boldsymbol{E} + \mu_1 t_1(x) + \cdots + \mu_d t_d(x)$$

is a versal deformation of $\boldsymbol{E}$.

Remark 5.23. *(Relation with preparation theorem)* In fact [Mat68, Prop. 3.6] allows us to reduce the problem to a finite dimensional one, in the setting of truncated formal power series. The algorithms given in Chaps. 6 and 7 can be interpreted as constructive proof of theorem 5.22 for formal power series. Mather proves that this implies theorem 5.22 for smooth functions. It is in this step that many technical difficulties are met, the solution of which requires Malgrange and Mather's preparation theorem (see also [Mar82, Ch. X]). As our focus is on the computation of a finite piece of the transformation, we shall not digress on this subject.

6 Gröbner bases and Standard bases

This chapter explains how to compute the codimension of the tangent spaces used in chapters 2 and 3: ideals, and left-right tangent spaces of the form (3.7), both as subsets of the ring of formal power series. For ideals, this can be done by calculating the formal power series equivalent of a Gröbner basis. This idea is generalized and applied to left-right tangent spaces.

6.1 Introduction

A Gröbner basis of an ideal in a polynomial ring is a set of generators with certain additional properties. One implication of these is that a normal form algorithm exists, equivalent to a 'division': Given an arbitrary polynomial, the algorithm expresses it as an ideal element, plus a unique remainder term in a certain minimal vector space. The dimension of this vector space is called the *codimension* of the ideal, which is finite for our application.

That application is Kas and Schlessinger's algorithm, constructing a right transformation inducing an arbitrary unfolding from a versal one. The ideal is related to the versal unfolding's tangent space, whereas the remainder terms are related to the (finitely many) deformation terms. Kas and Schlessinger's algorithm is the subject of Chap. 7.

These ideas need to be generalized somewhat, first of all because the tangent space we consider is not a *polynomial* ideal, but an ideal in the formal power series ring; see Chap. 5. Secondly, in the case of left-right transformations, the tangent space is no longer an ideal. For both of these cases, this chapter develops an appropriate normal form or 'division' algorithm.

Overview The problem of computing the codimension of an arbitrary polynomial ideal I in a polynomial ring R was solved by Bruno Buchberger in his 1965 thesis [Buc65]. His idea was to introduce an ordering on the monomials. The largest monomial occurring in a polynomial, with respect to this ordering, is called its *leading monomial*, and the ideal generated by the leading monomials of ideal members he called the *leading monomial ideal*, written as $\mathsf{LM}\, I$. The number of monomials not in $\mathsf{LM}\, I$ is precisely the codimension of I, i.e., the dimension of R/I as an $\mathbb{R}$-vector-space.

H. Broer, I. Hoveijn, G. Lunter, and G. Vegter: LNM 1806, pp. 97–132, 2003.

We now briefly sketch the main idea. Assume we have generators h_i of I, then clearly multiples of leading monomials of the h_i are in $\mathsf{LM}\, I$, but the converse is not generally true. Buchberger developed an algorithm computing a basis h_i' generating the same ideal I, but with the additional property that their leading monomials do generate $\mathsf{LM}\, I$. Such bases are called Gröbner bases. Gröbner bases are used to systematically solve a number of questions involving polynomial ideals. For some problems occurring in algebraic geometry, see [CLO98, GP88, MR88] and also [CLO92] which gives a good introduction to Gröbner basis theory. Many generalizations have been developed, for example for submodules (see [GP88]), for ideals in power series rings [Bec90a, Bec90b, Hir64, Mor88], and for subalgebras of polynomial rings [AHLM99, KM89, Mil96, RS90, Stu96, Vas98].

One problem in polynomial ideal theory which Gröbner bases solve, and which once was the main problem of the field (see van der Waerden [Wae60], and [Win96]), is the *ideal membership problem*: for a given f, decide whether it is an element of an ideal I. If a Gröbner basis of I is known, there is a *normal form algorithm*, which, for any equivalence class $f+I$ defined by a representative $f \in R$, returns a unique representative of that class. It is clear that this solves the ideal membership problem. A suitably modified version of the algorithm works in the ring of truncated formal power series, and is used in Chap. 7 to obtain reparametrizations and coordinate transformations related to versal unfoldings; see also [BHLV98, Lun99b].

In this chapter those generalizations are brought under a common umbrella. An abstraction is made both of of the base vector space (e.g. the ring of polynomials, truncated formal power series, or rational functions), and the algebraic structure of the set T of interest (ideal, submodule, subalgebra, left-right tangent space). Also the concept of 'monomial' is generalized: For our purposes the key property is not that monomials form an algebra, but that they form a basis of the base vector space (ring, module, algebra).

The algebraic structure is described by a map Ψ whose image is T, and we investigate its monomial structure. If this map satisfies certain properties, it is called a *standard map*. The main implication is that for such maps, the set $\mathsf{LM}\,\mathrm{Im}\,\Psi$ can be described explicitly, and using this, T's codimension can be computed.

The idea for this approach is based on the presentation of Greuel [GP88] for standard bases of submodules. More precisely, the proof of the standard map theorem 6.10 follows Greuel's proof of Schreyer's method for computing the module of syzygies of an ideal, stripped of the algebraic details unnecessary in the general setting. We also use the Schreyer order [Sch91] of monomials, which we call the *induced order* in the general context.

The algorithms for computing standard subalgebra bases, also known as *SAGBI*[1] or *canonical* bases, were taken from Sturmfels [Stu96]. The problems one encounters here are related to integer linear programming. Interestingly,

[1] *S*ubalgebra *A*nalogue of *G*röbner *B*ases for *I*deals, see e.g. [Vas98]

these can be solved using Gröbner bases again (see [CLO98, Ch. 8] and [Sch86, §16.4]). The algorithms to compute standard basis for left-right tangent spaces were based on these techniques, and also involve Gröbner basis calculations.

There are other possibilities for generalizations that are not mentioned yet, see e.g. [CCS99, Ch. 1]. In particular, we shall always assume that the coefficients are elements of a *field*, which avoids a number of complications that are encountered in the case of a base *ring*. For this topic we refer to e.g. [AHLM99, Mil96].

Organization of the chapter First we present the theory of Gröbner bases, without proofs, so as to suggest a generalization. In Sect. 6.3 the abstract setup is given, with the standard map theorem 6.10 underlying the subsequent results. Section 6.4 deals with several instances of standard bases, starting from Gröbner bases, and culminating in standard bases for left-right tangent spaces. The final section is about the differences encountered when these spaces live in the ring of (truncated) formal power series, instead of the polynomial ring.

6.1.1 Algorithms and real numbers

This chapter describes algorithms performing various computations. We here make some remarks how these algorithms are idealizations of their actual computer implementation. More down to earth, it can be regarded an attempt at justifying the use of real and complex numbers in the algorithm descriptions.

Mathematical models of computers, for example Turing machines (see e.g. [Dav65]), are usually discrete. This is reasonable, since modern computers are digital, and have well-defined discrete states. On the other hand, in mathematics we often use the fields $\mathbb{R}$ or $\mathbb{C}$ for computations, and their elements cannot be represented by a discrete model. So, when a mathematician wants to model algorithms performing 'real' calculations on a digital computer, there is a problem.

One solution would be to restrict to so-called *computable* fields, for instance finite fields, see e.g. [BW93]. However, one could also argue that the discreteness of digital computers is a detail, which should not, in this case, receive much emphasis. The numbers used by actual computers form a finite subset of the rationals, which however for most practical computations form a sufficiently dense subset of $\mathbb{R}$ to be a useful approximation of 'real' reals. A useful idealization of actual computers would then be a machine whose basic actions are conditional branches, and evaluation of formulas involving real (or complex) numbers. This is, very briefly, the point of view taken in [BSS89]; see also [Shu94, BCSS96] and the references there. With this in mind, we in this book present algorithms acting on ordinary real numbers, and consider this to be a reasonable idealization of the actual implementation.

6.2 Motivation: Gröbner bases

In this section we develop, without proofs, the notion of Gröbner bases for ideals. In the next section a generalization is given, which is presented along the same

lines. This section will step over many details, with the intention of suggestion a slightly different point of view towards Gröbner bases, rather than to prove the main theorems once more. An attempt at rigor will again be made from section 6.3 onwards.

6.2.1 Term orders for Gröbner bases

An important ingredient for Gröbner bases is the *term order*, an ordering of the monomials. To any polynomial $f \neq 0$, a monomial $\mathsf{LM}\, f$ is associated. It is called the *leading monomial*, and it is the largest monomial occurring in the polynomial, with respect to the term order. The coefficient associated to this monomial is called the *leading coefficient* (a real number, for now), and is denoted by $\mathsf{LC}\, f$. The product of these is the *leading term*: $\mathsf{LT}\, f = \mathsf{LC}(f)\, \mathsf{LM}(f)$.

The term orders that are used in this context, and the associated leading-monomial functions, have the following properties: (f, g, h nonzero polynomials)

a) The term order $\leq$ is a linear (or total) order, i.e., (1) it is transitive, (2) for any pair of monomials m, m' we have $m \leq m'$ or $m' \leq m$, and (3) if both hold then $m = m'$.
b) The term order $\leq$ is a well-order, i.e., every nonempty set of monomials has a smallest element.
c) $\mathsf{LM}\, f \leq \mathsf{LM}\, g \;\Leftrightarrow\; \mathsf{LM}(hf) \leq \mathsf{LM}(hg)$.
d) The set $\{\mathsf{LM}\, f | f \text{ a polynomial}\}$ forms a basis of the polynomial ring.
e) $\mathsf{LM}(f - \mathsf{LT}\, f) < \mathsf{LM}\, f$.
f) $\mathsf{LM}(f - g) \leq \max(\mathsf{LM}\, f, \mathsf{LM}\, g)$, and equality holds unless $\mathsf{LT}\, f = \mathsf{LT}\, g$.

These properties are not independent. Later we shall extend the notion of 'monomial' and 'monomial order', and require only some of the properties above to hold. For convenience later on, we set $\mathsf{LM}\, 0 = \mathsf{LT}\, 0 = 0$, and also $0 < m$, for all monomials m.

Examples For one variable there is only one term-order: $1 < x < x^2 < \cdots$. With more variables it becomes more interesting. The *lexicographic order*, or *lex-order* for short, symbolically $<_{\text{lex}}$, is defined by $x^\alpha <_{\text{lex}} x^\beta$ iff the left-most nonzero entry in $\alpha - \beta \in \mathbb{Z}^n$ is negative;[2] for example $x_1^2 x_2^{10} <_{\text{lex}} x_1^3$, but $x_1^3 x_2^{10} >_{\text{lex}} x_1^3$. This order is an example of an *elimination order* (for x_n): A Gröbner basis of an ideal with respect to this term order contains a polynomial from which the variables $x_1, \ldots, x_{n-1}$ are eliminated, if the ideal contains such elements at all.

A direct Gröbner basis calculation using the lexicographic order can take much time. *Graded* term orders perform much better. This is especially true for the *graded reverse lexicographic order*, or *grevlex* for friends. It is defined by $x^\alpha <_{\text{grevlex}} x^\beta$ iff $\deg(x^\alpha) < \deg(x^\beta)$, or $\deg(x^\alpha) = \deg(x^\beta)$ and the right-most nonzero entry in $\alpha - \beta$ is positive. (Here $\deg(x^\alpha) = \alpha_1 + \cdots + \alpha_n$.)

[2] Here we use the multi-index notation $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

There exist $n!$ variants of the lex- and grevlex-orders, obtained by permuting the variables. Many more term orders satisfying the properties (a) to (f) above exist, and a full classification is given in appendix A.1.

6.2.2 Basic question

The defining property of a Gröbner basis can be expressed using the leading-monomial function LM. Let h_i be some polynomials, and let $I = \langle h_i \rangle$ be the associated ideal. Let $\mathsf{LM}\, I := \mathrm{span}_{\mathbb{R}}\{\mathsf{LM}\, f | f \in I\}$ be the linear span of all leading monomials of ideal members. The set $\mathsf{LM}\, I$ is in general difficult to describe. On the other hand, the ideal $\langle \mathsf{LM}\, h_1, \ldots, \mathsf{LM}\, h_k \rangle$ is a monomial ideal with explicit generators. This ideal is easy to work with, for example the membership problem is trivial.

In general $\langle \mathsf{LM}\, h_1, \ldots, \mathsf{LM}\, h_k \rangle \subseteq \mathsf{LM}\, \langle h_1, \ldots, h_k \rangle$. We are interested in the following question: Given a set of generators $\{h_1, \ldots, h_k\}$ of an ideal, under what conditions is it true that

$$\mathsf{LM}\, \langle h_1, \ldots, h_k \rangle = \langle \mathsf{LM}\, h_1, \ldots, \mathsf{LM}\, h_k \rangle \quad ? \tag{6.1}$$

Bases for which equality holds are called Gröbner bases. It is not difficult to prove that Gröbner bases exist for any ideal (of a Noetherian ring, to be precise). A very natural question to ask is: Given a set of generators $\{h_1, \ldots, h_m\}$ for an ideal I, how can one modify this set of generators such that they still generate I but at the same time also satisfy (6.1)? The algorithm accomplishing this is the *Buchberger algorithm*.

6.2.3 Rephrasing the basic question

We now put the basic question (6.1) in a different form. At this point it is convenient to introduce some notation. Let R be the base ring. For Gröbner bases we use $R = \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$, the polynomial ring over $\mathbb{R}$ in n variables.

Consider the diagram of Fig. 6.1. In this diagram M is the free module $R^k = \oplus_{i=1}^k R e_i$, and $e_i = (0, \ldots, 1, \ldots, 0)$ denotes the i-th basis vector in R^k. The map Ψ is an R-module homomorphism mapping e_i to h_i, so that the image of Ψ is the ideal $I = \langle h_1, \ldots, h_k \rangle$. The map $\widetilde{\Psi}$ is also an R-module homomorphism, but this one maps e_i to $\mathsf{LT}\, h_i$. By construction, therefore, the image of $\widetilde{\Psi}$ is contained in $\mathsf{LM}\, I = \mathsf{LT}\, I$. The basic question posed in the previous section can now be rephrased as: Under what conditions do we have that

$$\mathrm{Im}\, \widetilde{\Psi} = \mathsf{LM}\, \mathrm{Im}\, \Psi \quad ? \tag{6.2}$$

Note that $\mathsf{LM}\, \mathrm{Im}\, \Psi \neq \mathrm{Im}(\mathsf{LT}\, \Psi)$: The map $\mathsf{LT}\, \Psi$ is not a linear map, and the diagram of Fig. 6.1 never commutes.

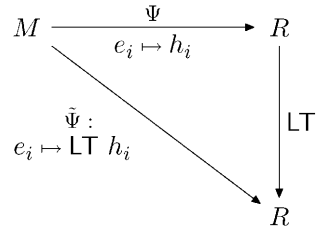

Fig. 6.1 Gröbner basis diagram

6.2.4 A criterion for Gröbner bases

In order to formulate the criterion implying equality in (6.2), we need a few more definitions.

Monomials First of all we extend the notion of *monomial* to M. A monomial in M is an element of the form $x^\beta e_i$ for some $\beta \in \mathbb{N}^n$ and $1 \leq i \leq k$. A *term* is a monomial multiplied by a coefficient.

Standard representations Assume that $\{h_1, \ldots, h_k\}$ is not a Gröbner basis and hence that equality fails in (6.2). In other words, for a certain $\alpha \in M$ we have $\mathsf{LM}\,\Psi\alpha \notin \operatorname{Im}\widetilde{\Psi}$. Consider the set $\{\mathsf{LM}\,\psi t\}$ where t runs over the terms of α. The highest monomial of these is not equal to $\mathsf{LM}\,\Psi\alpha$, as otherwise the latter would be an element of $\operatorname{Im}\widetilde{\Psi}$. In other words, summing all Ψt cancels this highest monomial.

We say that an $\alpha \in M$ is a *standard representation* (of $\Psi\alpha$) if such cancellation of the leading monomial does *not* occur. This can be expressed as: $\mathsf{LM}\,\Psi t \leq \mathsf{LM}\,\Psi\alpha$ for all terms t of α. It is easy to see that every $f \in I = \operatorname{Im}\Psi$ has a standard representation, if and only if $\{h_1, \ldots, h_k\}$ is a Gröbner basis.

Remark 6.1. Our notion of *standard representation* is the natural translation to the current set-up of a similar notion in [BW93, Def. 5.59].

Division; normal form Consider the following algorithm. It is a generalization of the polynomial division algorithm to the case of multiple polynomials. Given an input element f, it expresses f as a member of the ideal $\langle h_1, \ldots, h_k\rangle$ plus a remainder term r, subject to some conditions:

Algorithm 6.2. *(Normal form)*
Input: $f, h_1, \ldots, h_k \in R$
Output: $\alpha \in M,\ r \in R$ *such that*

1. $f = r + \Psi\alpha$
2. $r = 0$ *or* $\mathsf{LM}\,r \notin \operatorname{Im}\widetilde{\Psi}$
3. α *is a standard representation.*

Algorithm:

$\alpha \leftarrow 0$
$r \leftarrow f$
While $r \neq 0$ *and* $\mathsf{LT}\, r = cx^\beta\, \mathsf{LT}(h_i)$ *for some* c, β, i, *do:*
    $\alpha \leftarrow \alpha + cx^\beta e_i$
    $r \leftarrow r - cx^\beta h_i$
EndWhile

Proof: The equality $f = r + \Psi\alpha$ is an invariant of the While-loop: The condition of the While-body implies the second statement, and α is a standard representation because if $\alpha = t_1 + t_2 + \ldots$, where the t_i denote the terms of α in the order in which they were added, then $\mathsf{LM}\,\Psi t_i$, $i = 1, 2, \ldots$ is a strictly decreasing sequence of monomials. Termination, finally, is guaranteed because $\mathsf{LM}\, r$ is strictly decreasing, and the term order is a well-order. ∎

Now if $f \in I$ and $\{h_1, \ldots, h_k\}$ is a Gröbner basis, the algorithm will terminate with $r = 0$ because at each stage $r \in I$ and therefore $\mathsf{LM}\, r \in \operatorname{Im} \widetilde{\Psi} = \langle \mathsf{LM}\, h_1, \ldots, \mathsf{LM}\, h_k \rangle$. In other words, if $\{h_1, \ldots, h_k\}$ is a Gröbner basis the algorithm yields a standard representation for any ideal element. Conversely, if $r = 0$ for any $f \in I$ then $\{h_1, \ldots, h_k\}$ must be a Gröbner basis.

S-polynomials The conclusion of the previous section can be rephrased as: If for any $\alpha \in M$ which is *not* a standard representation, there exists a standard representation β such that $\Psi\alpha = \Psi\beta$, then $\{h_1, \ldots, h_k\}$ is a Gröbner basis.

So how do we find those $\alpha \in M$ which are not standard representations? For such α we have "cancellation of leading monomials". More precisely, let $\alpha = t_1 + t_2 + \cdots$, let $m = \max_i \mathsf{LM}(\Psi t_i) = \widetilde{\Psi} t_p$, and assume the t_i form a non-increasing sequence: $\widetilde{\Psi} t_i = m$ for $i \leq p$, and $\widetilde{\Psi} t_i < m$ for $i > p$. Since α is not a standard representation, $\mathsf{LM}\,\Psi\alpha < m$. As $t_1, \ldots, t_p$ are the only terms involving m this implies $\mathsf{LM}\,\Psi(t_1 + \cdots + t_p) < m$, that is, $\widetilde{\Psi}(t_1 + \cdots + t_p) = 0$. This gives some motivation as to why $\ker \widetilde{\Psi}$ might be of interest. (Note however that α itself need not be an element of $\ker \widetilde{\Psi}$.)

Generators s_{ij} of $\ker \widetilde{\Psi}$ as a module over R are easy to give explicitly:

$$s_{ij} = \frac{\mathsf{LT}\, h_j}{\gcd(\mathsf{LM}\, h_i, \mathsf{LM}\, h_j)} e_i - \frac{\mathsf{LT}\, h_i}{\gcd(\mathsf{LM}\, h_i, \mathsf{LM}\, h_j)} e_j, \qquad (1 \leq i < j \leq k) \tag{6.3}$$

and the images Ψs_{ij} are the well-known S-polynomials, see e.g. [CLO92].

Remark 6.3. *(Binomial kernel)* Note that the s_{ij} are binomials. This is related to the fact that $\widetilde{\Psi}$ is a *monomial mapping*: a mapping that maps monomials to monomials. See also Definition 6.8.

Remark 6.4. *(Buchberger's criteria)* The set of generators (6.3) is not a minimal set. Buchberger's first and second criterion (see e.g. [BW93, CLO92]) may be interpreted as sufficient conditions for generators to be superfluous.

Gröbner bases and Buchberger's algorithm Recall the basic question: Under which condition is $\operatorname{Im}\widetilde{\Psi} = \mathsf{LM}\operatorname{Im}\Psi$. The answer is now easy to give:

Theorem 6.5. *(Gröbner basis) If algorithm 6.2 gives output $r = 0$ on input $f = \Psi(s_{ij})$ for all generators s_{ij} in (6.3), then $\{h_1, \ldots, h_k\}$ is a Gröbner basis.*

Buchberger's algorithm consists of checking the criterion, and adding the nonzero r, which lies in the ideal $\operatorname{Im}\Psi$ by definition of algorithm 6.2, to the ideal generators until the criterion holds. At each step the ideal $\operatorname{Im}\widetilde{\Psi}$ increases. Since the polynomial ring is Noetherian, an increasing chain of ideals stabilizes, implying termination of this algorithm.

6.3 Standard bases

6.3.1 Overview

In this section we put the previous discussion in a more abstract setting. One advantage is that this makes the proof of the Gröbner basis case more transparent. A more important advantage is that it allows for generalizations, in various directions.

One direction is changing the base ring, from the polynomial ring to the ring of formal power series, and later to truncated formal power series. This is of interest to us because formal power series arise naturally from the Birkhoff normal form procedure. Related to the change of base ring is the introduction of non-well-orders for the term orders. The notion of 'leading term' is undefined for formal power series if the term order regards monomials with large exponents as large. The solution is to 'flip' the term order and to regard 1 as the highest monomial. This destroys the well-orderedness however.

A second direction to generalize in is to allow other algebraic structures than ideals, the objects ordinary Gröbner bases deal with. A generalization to modules is well-known, see e.g. [GP88]. Another generalization, for subalgebras, is known as a SAGBI basis [Vas98], or canonical subalgebra basis as Sturmfels [Stu93, Stu96] calls it. For our purposes we need the analogous basis for still different vector spaces. This section gives the basic set-up for all these cases. We formulate the *standard map theorem*, which lies at the heart of all generalizations of Gröbner basis mentioned. In later sections we specialize to several cases of interest.

6.3.2 Definitions

A few details in the definition of term order we use, is different from the usual definition for Gröbner bases; some others, like the Schreyer ordering, need to be generalized. Here we collect the necessary definitions.

Base field The field of coefficients is denoted by $\mathbf{R}$; see also Sect. 6.1.1 for remarks. Note that it is sometimes useful to compute over a coefficient *ring*, see e.g. [AHLM99, Mil96]; we shall not use this.

Term order and monomials Let M be some vector space over $\mathbf{R}$. Already in the previous section we used the name 'monomial' also for elements of the form $x^\beta e_i$, which formed a basis of the free module R^k. Presently we want to be even more general. It is not possible to be very specific about the term 'monomial' here, because we do not want to be specific about the vector space M. For example, for Gröbner bases M is a module, but for canonical subalgebra bases M would be a ring. So instead, we just suppose a set of monomials has been defined, together with a term order. Also we suppose the functions LM and LC, for *leading monomial* and *leading coefficient* are defined on M.

The functions $\mathsf{LM} : M \to M$ and $\mathsf{LC} : M \to \mathbf{R}$ are required to have the following basic properties, for all $f \in M$ and $a \in \mathbf{R}$:

1. $\mathsf{LM}\,\mathsf{LM}\,f = \mathsf{LM}\,f$ (provided $f \neq 0$)
2. $\mathsf{LC}\,\mathsf{LM}\,f = 1$ (provided $f \neq 0$)
3. $\mathsf{LM}\,af = \mathsf{LM}\,f$ (provided $a \neq 0$)
4. $\mathsf{LC}\,af = a\,\mathsf{LC}\,f$

The leading term, denoted by LT, is defined as $\mathsf{LT}\,f = \mathsf{LC}\,f \cdot \mathsf{LM}\,f$. The ordering of the monomials is extended to the terms by simply ignoring the coefficient. (However, if we write $t = t'$ we mean that $t - t' = 0$, instead of just $t \leq t'$ and $t' \leq t$ with respect to the term order.) The other properties we require of the leading-monomial function (and the related term order) are:

a) The set $\mathrm{span}_{\mathbf{R}}\{\mathsf{LM}\,f | f \in M\}$ is dense in M.
b) The term order $\leq$ is a linear (or total) order: For any pair of monomials m, m' we have $m \leq m'$ or $m' \leq m$, and if both hold then $m = m'$, and the term order is transitive.
c) $\mathsf{LM}(f - g) \leq \max(\mathsf{LM}\,f, \mathsf{LM}\,g)$, and equality holds unless $\mathsf{LT}\,f = \mathsf{LT}\,g \neq 0$.

For convenience we also set $\mathsf{LM}\,0 = 0$. A few remarks are in order:

– Generally, M is an infinite dimensional $\mathbf{R}$-vector space.
– Property (c), (1) and (3) together imply that

$$\mathsf{LM}(f - \mathsf{LT}\,f) < \mathsf{LM}\,f, \qquad (f \neq 0) \tag{6.4}$$

– From (4) and (c) it follows that $\mathsf{LT}\,0 = 0$, and $0 < m$ for all (nonzero) monomials m.
– The term order is not required to be a well-order.
– The leading-monomial function is not required to be multiplicative.

In relation to the last remark, note that M need not be a ring, or even a module over some ring, so that it is not clear what 'multiplicative' should mean. This

level of generality is only needed here, and is enough to formulate the main result of this section, the standard map theorem. In the applications of Sect. 6.4 we always require the term order to be multiplicative, in some appropriate sense.

For a linear subspace L of M, we write $\mathsf{LM}\, L$ for the closure of $\mathrm{span}_{\mathbf{R}}\{\mathsf{LM}\, f | f \in L\}$. For example, we have $M = \mathsf{LM}\, M$. Another example, if M is a ring and L an ideal, then $\mathsf{LM}\, L$ is the leading monomial ideal, also called the *initial ideal*.

Refined term orders Suppose we have a (linear) map $\Psi : L \to M$, and term orders on L and M. For clarity we write LM_L and LM_M for the leading-monomial functions on L and M respectively. The term order on L is said to be a *refinement* of the one on M (via Ψ, if we want to be precise), if

$$\mathsf{LM}_M \Psi\alpha \;\leq_M\; \mathsf{LM}_M \Psi \mathsf{LM}_L \alpha \tag{6.5}$$

for all $\alpha \in L$. Using the basic properties, this implies that for all monomials $m, m' \in L$ we have

$$\mathsf{LM}_M\ \Psi m <_M \mathsf{LM}_M\ \Psi m' \quad\Rightarrow\quad m <_L m', \tag{6.6}$$

$$m <_L m' \quad\Rightarrow\quad \mathsf{LM}_M \Psi m \;\leq_M\; \mathsf{LM}_M \Psi m', \tag{6.7}$$

which motivates the name *refinement*. If L is a free module over M, the refined term order on L is called the Schreyer order, see [GP88, Sch91].

Assumption From here on, it will be assumed that whenever there is a map $\Psi : L \to M$ and a term order on M, there is also a term order on L which is a refinement, via Ψ, of the order on M. Moreover we will just write LM instead of LM_L or LM_M; which one is intended will be clear from the context.

Standard representations Using the refined term orders, it is possible to give an elegant definition of a standard representation (see Sect. 6.2.4). Again assume we have a map $\Psi : M \to R$, then an element $\alpha \in M$ is said to be a standard representation if cancellation of leading monomials does not occur:

Definition 6.6. *An element $\alpha \in M$ is called a* standard representation *(of $\Psi\alpha$) if*

$$\mathsf{LM}\, \Psi\alpha = \mathsf{LM}\, \Psi\, \mathsf{LM}\, \alpha.$$

6.3.3 Setup

Now we formulate the analogue of the basic question posed in section 6.2.2 in the current context. Assume we have a linear map $\Psi : M \to R$, and we wish to know its image. In particular, we are interested in the set of leading monomials that occur in the image of Ψ.

Related to Ψ is a map $\widetilde{\Psi}$ (see Fig. 6.2). On monomials $m \in M$ it is defined as $\widetilde{\Psi} m := \mathsf{LT}\, \Psi m$, and it is extended to all of M by linear combinations and closure. $\widetilde{\Psi}$ is a monomial map, and it is easy to describe its range. Moreover

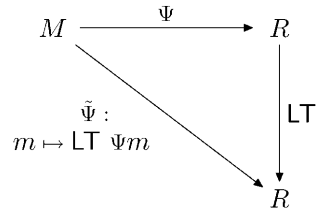

Fig. 6.2 Standard maps and bases

it is clear that $\operatorname{Im}\widetilde{\Psi} \subset \mathsf{LM}\operatorname{Im}\Psi$. The problem is to determine whether in fact $\operatorname{Im}\widetilde{\Psi} = \mathsf{LM}\operatorname{Im}\Psi$. When this is true, Ψ is called a *standard map*. (Usually Ψ is defined in terms of some basis, which is then called a *standard basis*.)

A related problem is to find the kernel of Ψ. In the case where $\operatorname{Im}\Psi$ is the ideal $\langle h_1, \ldots, h_k \rangle$, this is the question of finding the so-called *syzygies* for the ordered set of polynomials $\{h_1, \ldots, h_k\}$, i.e., k-tuples of polynomials $(a_1, \ldots, a_k)$ such that

$$a_1 h_1 + \cdots + a_k h_k = 0.$$

It is easy to see that the set of these k-tuples indeed forms an R-module. The standard map theorem below gives a relation between the kernel of Ψ and the kernel of $\widetilde{\Psi}$, which in the case of ideals boils down to a relation between syzygies of the set $\{h_1, \ldots, h_k\}$ and those of the set $\{\mathsf{LM}\, h_1, \ldots, \mathsf{LM}\, h_k\}$. The latter module is generated by the (pre-images of the) S-polynomials (6.3).

Remark 6.7. *(Syzygies)* Below we shall use the word *syzygy* to refer to syzygies on *leading monomials* of generators, i.e., elements of $\ker\widetilde{\Psi}$ instead of elements of $\ker\Psi$.

6.3.4 Normal form property

In the case of Gröbner bases, that is, a well-order and a polynomial ring, we used algorithm 6.2 to bring an arbitrary element of the polynomial ring into a normal form. The analogue of the algorithm can be written down in the current general context, but will in general not terminate. So instead of writing down a normal form algorithm, we assume existence of a *normal form map* with the necessary properties. The normal form map that we use here is very similar to the one used by Greuel [GP88], with the difference that he does not consider the α–part – see below.

Assume we have the situation of Fig. 6.2. A *normal form map* for $\Psi : M \to R$ is a map

$$\mathrm{NF}^{\Psi} : \quad R \to M \oplus R : \quad f \mapsto \mathrm{NF}^{\Psi}(f) \;=\; \big(\mathrm{NF}^{\Psi}_{\alpha}(f), \mathrm{NF}^{\Psi}_{r}(f)\big)$$

with the following properties (we write $r = \mathrm{NF}^{\Psi}_{r}(f)$, $\alpha = \mathrm{NF}^{\Psi}_{\alpha}(f)$):

a) $f = \Psi\alpha + r$,
b) $r = 0$ or $\mathsf{LM}\, r \notin \operatorname{Im} \widetilde{\Psi}$,
c) α is a standard representation.

Heuristically, the map NF^{Ψ} performs a *division*, with α the 'quotient' and r the 'remainder'. The map Ψ is said to have the *normal form property* if a normal form map NF^{Ψ} exists. This normal form property holds under very general (topological) conditions; for example it always exists in polynomial rings with a well-order as term order.

6.3.5 The standard map theorem

The following theorem is inspired on the proof of the standard basis theorem (for ideals in the local polynomial ring, with non-well-orders) by Greuel [GP88], which he in turn attributes partly to Schreyer [Sch91].

Definition 6.8. *A linear map $\widetilde{\Psi}$ is called a* monomial map *if $\widetilde{\Psi} m$ is a monomial for every monomial m.*

Given a map Ψ, its associated monomial map $\widetilde{\Psi}$ is defined on the set of monomials m by $\widetilde{\Psi} m := \mathsf{LT}\, \Psi m$, and extended linearly for other elements. This is a good definition if $\Psi m \neq 0$ for all monomials m, which will be assumed. Now recall the definition of *standard map*:

Definition 6.9. *Let $\widetilde{\Psi}$ be the monomial map associated to Ψ. The map Ψ is called a* standard map *if*

$$\mathsf{LM} \operatorname{Im} \Psi = \operatorname{Im} \widetilde{\Psi}.$$

Theorem 6.10. *(Standard map theorem) Let L, M, R be vector spaces, and $\Phi : L \to M$ and $\Psi : M \to R$ linear maps, having associated monomial maps $\widetilde{\Phi}$ and $\widetilde{\Psi}$, and assume they have the following properties:*

a) $\operatorname{Im} \Phi \subseteq \ker \Psi$,
b) $\operatorname{Im} \widetilde{\Phi} \supseteq \mathsf{LM} \ker \widetilde{\Psi}$,
c) $\Phi : L \to M$ *has the normal form property.*

Then:

1. $\operatorname{Im} \Phi = \ker \Psi$,
2. $\operatorname{Im} \widetilde{\Psi} = \mathsf{LM} \operatorname{Im} \Psi$,
3. $\operatorname{Im} \widetilde{\Phi} = \mathsf{LM} \operatorname{Im} \Phi$.

In other words, the conclusion of the theorem is that $L \xrightarrow{\Phi} M \xrightarrow{\Psi} R$ is an exact sequence, and Φ and Ψ are standard maps. When applying the theorem, one constructs Φ such that condition (a) is satisfied. Condition (b) is easy to check, as it involves only monomial maps. It corresponds to the Gröbner basis criterion (Theorem 6.5) in the case of polynomial ideals.

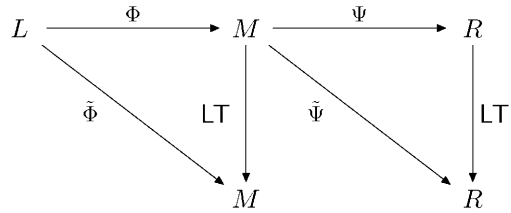

Fig. 6.3 Diagram for the standard map theorem

Proof: Choose any $\beta \in M$, and let $f := \Psi\beta$ be its image in R. Let NF^{Φ} be a normal form for Φ (which exists by assumption (c)), and set $A := \mathrm{NF}^{\Phi}_{\alpha}(\beta)$ and $\rho := \mathrm{NF}^{\Phi}_{r}(\beta)$. Then we have:

$$\beta = \Phi A + \rho, \qquad \rho = 0 \text{ or } \mathsf{LM}\,\rho \notin \operatorname{Im}\widetilde{\Phi}. \tag{6.8}$$

By assumption (a) we have $\Phi A \in \ker\Psi$. Therefore

$$f = \Psi\beta = \Psi(\Phi A + \rho) = \Psi\rho.$$

We now claim: Either $f \neq 0$ and $\mathsf{LM}\,f = \mathsf{LM}\,\Psi\,\mathsf{LM}\,\rho$, or $\rho = 0$. To prove this, assume $\rho \neq 0$ and define

$$\begin{aligned} t_1 &:= \mathsf{LT}\,\rho, \\ \rho' &:= \rho - t_1, \\ t_2 &:= \mathsf{LT}\,\rho'. \end{aligned}$$

Then $\mathsf{LM}\,t_1 > \mathsf{LM}\,t_2 = \mathsf{LM}\,\rho'$, and using (6.5) and (6.7) we get $\mathsf{LM}\,\Psi t_1 \geq \mathsf{LM}\,\Psi t_2 \geq \mathsf{LM}\,\Psi\rho'$. Now if $\mathsf{LM}\,\Psi t_1 > \mathsf{LM}\,\Psi t_2$, then $\mathsf{LM}\,\Psi\rho = \mathsf{LM}(\Psi t_1 + \Psi\rho') = \mathsf{LM}\,\Psi t_1 = \mathsf{LM}\,\Psi\,\mathsf{LM}\,\rho$, proving our claim. So assume on the contrary that $\mathsf{LM}\,\Psi t_1 = \mathsf{LM}\,\Psi t_2$. In particular $\mathsf{LC}\,\Psi t_2 \neq 0$, hence also $\mathsf{LC}\,\widetilde{\Psi} t_2 \neq 0$. Define

$$t_2' := \frac{\mathsf{LC}\,\widetilde{\Psi} t_1}{\mathsf{LC}\,\widetilde{\Psi} t_2} t_2.$$

Then $\widetilde{\Psi} t_1 = \widetilde{\Psi} t_2'$, that is, $t_1 - t_2' \in \ker\widetilde{\Psi}$. Taking the LM on both sides yields $\mathsf{LM}\,t_1 \in \mathsf{LM}\ker\widetilde{\Psi}$. Since $\mathsf{LM}\,t_1 = \mathsf{LM}\,\rho$ and $\mathsf{LM}\ker\widetilde{\Psi} \subseteq \operatorname{Im}\widetilde{\Phi}$ (assumption (b)), this implies $\mathsf{LM}\,\rho \in \operatorname{Im}\widetilde{\Phi}$, contradicting (6.8).

So indeed $f \neq 0$ and $\mathsf{LM}\,f = \mathsf{LM}\,\Psi\,\mathsf{LM}\,\rho = \widetilde{\Psi}\,\mathsf{LM}\,\rho$, or $\rho = 0$. The first part proves that $\mathsf{LM}\operatorname{Im}\Psi \subseteq \operatorname{Im}\bar{\Psi}$. As the other inclusion is trivial, this proves (2). The second part says that if $\beta \in \ker\Psi$ then $\rho = \mathrm{NF}^{\Phi}_{r}(\beta) = 0$, that is, $\beta = \Phi A$. In other words $\ker\Psi \subseteq \operatorname{Im}\Phi$. Together with (a) this proves (1). Finally, using (1) we get that $\mathrm{NF}^{\Phi}_{r}(\beta) = 0$ for all $\beta \in \operatorname{Im}\Phi$, so that every such β has a standard representation, implying (3). ∎

The following two lemmas will be helpful in applying the standard map theorem 6.10.

Binomial lemma In the case of Gröbner bases, the kernel of $\widetilde{\Psi}$ is generated by binomials, which correspond to the S-polynomials. This holds more generally. The following lemma is a generalization of Lemma 4.1 of [Stu96]. The proof is different, since we cannot use the well-ordering property of the term order. (Recall that a *term* is a monomial multiplied by a constant.)

Lemma 6.11. *(Binomial generators) Let $\widetilde{\Psi}$ be a monomial map. Then* $\ker \widetilde{\Psi}$ *is the closure of*

$$\operatorname{span}_{\mathbf{R}}\{t - t' | t, t' \text{ terms, and } \widetilde{\Psi}t = \widetilde{\Psi}t'\}. \tag{6.9}$$

Proof: First define $\widehat{R} := \{\mathsf{LM}\, f | f \in R\}$, and $\widehat{M} := \{\mathsf{LM}\, f | f \in M\}$. For every $r \in \widehat{R}$ select an $m_r \in \widehat{M}$ such that $\widetilde{\Psi}m_r = c \cdot r$ (if one exists), for some $c \in \mathbf{R}$. Now let $\alpha \in \ker \widetilde{\Psi}$. Using that M is the closure of $\operatorname{span}_{\mathbf{R}} M$, we may write $\alpha = \sum_{m \in \widehat{M}} c_m m$. Define

$$\alpha' := \sum_{m \in \widehat{M}} \left(c_m m - c_m \frac{\mathsf{LC}\, \widetilde{\Psi}m}{\mathsf{LC}\, \widetilde{\Psi}m_{\mathsf{LM}\, \widetilde{\Psi}m}} m_{\mathsf{LM}\, \widetilde{\Psi}m} \right).$$

Write $r = \mathsf{LM}\, \widetilde{\Psi}m$, then

$$\widetilde{\Psi}\left(c_m \frac{\mathsf{LC}\, \widetilde{\Psi}m}{\mathsf{LC}\, \widetilde{\Psi}m_r} m_r \right) = c_m \frac{\mathsf{LC}\, \widetilde{\Psi}m}{\mathsf{LC}\, \widetilde{\Psi}m_r}\, \mathsf{LC}(\widetilde{\Psi}m_r)\, \mathsf{LM}(\widetilde{\Psi}m_r) = c_m \mathsf{LC}(\widetilde{\Psi}m) r = \widetilde{\Psi}(c_m m),$$

since $\mathsf{LM}\, \widetilde{\Psi}m_r = r$. This shows that α' is in the closure of (6.9). For any $r \in \widehat{R}$, the sum of the coefficients $c_m\, \mathsf{LC}\, \widetilde{\Psi}m$ over all m such that $m_{\mathsf{LM}\, \widetilde{\Psi}m} = m_r$ vanishes, since this is precisely the coefficient of r in $\widetilde{\Psi}\alpha$. But this means that $\alpha' = \alpha$, which completes the proof. ∎

Standard representations of syzygies The following lemma asserts that syzygies cannot be standard representations:

Lemma 6.12. *Let $t_1 - t_2$ be a binomial in the kernel of $\widetilde{\Psi}$, and let α be a standard representation of $\Psi(t_1 - t_2)$. Then $t_1 > \mathsf{LM}\, \alpha$.*

Proof: Since $t_1 - t_2 \in \ker \widetilde{\Psi}$ we have $\mathsf{LT}\, \Psi t_1 = \mathsf{LT}\, \Psi t_2$. By (6.4) it follows that $\mathsf{LM}\, \Psi t_1 > \mathsf{LM}\, \Psi(t_1 - t_2) = \mathsf{LM}\, \Psi\alpha = \mathsf{LM}\, \Psi\, \mathsf{LM}\, \alpha$, the last equality holding since α is a standard representation. By (6.6), this implies $t_1 > \mathsf{LM}\, \alpha$. ∎

6.3.6 Normal form algorithm

It often happens that a map Ψ not only has the normal form property (see Sect. 6.3.4), but there even exists an algorithm that computes this normal form. Often this algorithm is the following. Note that when h_i are polynomials and Ψ is the module homomorphism $(f_1, \ldots, f_r) \mapsto \sum_i h_i f_i$ into the polynomial ring, then the algorithm below is just algorithm 6.2.

Algorithm 6.13. *(Normal form)*
Input: *Map* $\Psi : M \to R$ *and associated monomial map* $\widetilde{\Psi}$.
Output: $\alpha \in M,\ r \in R$ *such that*

1. $f = r + \Psi\alpha$,
2. $r = 0$ *or* $\mathsf{LM}\, r \notin \operatorname{Im} \widetilde{\Psi}$,
3. α *is a standard representation.*

Algorithm:

$\alpha \leftarrow 0$
$r \leftarrow f$
While $\mathsf{LT}\, r \in \operatorname{Im} \widetilde{\Psi}$, *say* $\mathsf{LT}\, r = \widetilde{\Psi} t$, t *a term, do:*
    $\alpha \leftarrow \alpha + t$,
    $r \leftarrow r - \Psi t$
EndWhile

In general it may be nontrivial to decide whether a term $\mathsf{LT}\, r$ is in the image of $\widetilde{\Psi}$ or not, and to find a term t in the pre-image.

Correctness of algorithm 6.13 is straightforward, but termination less so. If algorithm 6.13 does not terminate, there may exist other algorithms that do. One example is Mora's normal form for the rational function ring; see [GP88, Mor82, Mor85].

6.3.7 Reduced normal forms

For computations it is sometimes useful to use a more restricted notion of normal form, the *reduced normal form*. Indeed, the algorithm of this section will find application in Chap. 7. Whereas the 'remainder' $r = \mathrm{NF}_r^{\Psi}$ of an ordinary normal form need only satisfy $\mathsf{LM}\, r \notin \operatorname{Im} \widetilde{\Psi}$, for a reduced normal form this is required, not only of the leading monomial, but of *all* terms of r. In practice this means that computing such reduced normal forms is less efficient compared to ordinary normal forms. The big asset of reduced normal forms is that the r-part of their output is *unique*, when computed relative to a standard basis.

It is not true that the normal form property also implies the existence of a *reduced* normal form: in the rational function ring the Mora normal form exists, which cannot be extended to a reduced normal form. However, in many cases a reduced normal form does exist, and the algorithm below usually suffices:

Algorithm 6.14. *(Reduced normal form)*
Input: *Map* $\Psi : M \to R$ *and associated monomial map* $\widetilde{\Psi}$.
Output: $\alpha \in M,\ r \in R$ *such that*

1. $f = r + \Psi\alpha$
2. $r = \sum_{i \in I} t_i$ *with* t_i *terms such that* $t_i \notin \operatorname{Im} \widetilde{\Psi}$
3. α *is a standard representation.*

Algorithm:

$\alpha \leftarrow 0$
$r \leftarrow 0$
$g \leftarrow f$
While $g \neq 0$ *do the following:*
    If $\mathsf{LT}\, g \in \operatorname{Im} \widetilde{\Psi}$, *say* $\mathsf{LT}\, g = \widetilde{\Psi} t$, t *a term, then:*
        $\alpha \leftarrow \alpha + t$,
        $g \leftarrow g - \Psi t$
    Else:
        $r \leftarrow r + \mathsf{LT}\, g$
        $g \leftarrow g - \mathsf{LT}\, g$
    EndIf
EndWhile

6.4 Instances of standard bases

In this section we apply the previous theory to some known cases, such as Gröbner bases and canonical subalgebra bases. This shows how the current approach unifies some other approaches. The standard map theorem is also used to define the concept of standard basis for a left-right tangent space.

6.4.1 Gröbner bases

Let $\{h_1, \ldots, h_k\}$ be a set of polynomials in $R := \mathbf{R}[x_1, \ldots, x_n] = \mathbf{R}[x]$. Let M be the free R-module $R^k := \oplus_{i=1}^k R e_i$. The R-module homomorphism $\Psi : M \to R$ is defined by $\Psi e_i = h_i$, so that $\operatorname{Im} \Psi = \langle h_1, \ldots, h_k \rangle$. See Fig. 6.1.

For monomials in R we take the ordinary monomials, and in R^k we take the elements of the form $e_i x^\beta$, where e_i is the i-th canonical basis vector. For the term order (on M and R, with the one on M being a refinement via Ψ of term order on R) we take a general well-order, but one which is multiplicative over R, that is, $m < m'$ implies $\widetilde{m}m < \widetilde{m}m'$ for all monomials $m, m', \widetilde{m}$. Then the map $\widetilde{\Psi}$, defined in the general way by $\widetilde{\Psi} m := \mathsf{LT}\, \Psi m$ for monomials $m \in M$, is in fact an R-module homomorphism.

Algorithm 6.2 implements a normal form NF^Ψ on R, so that the normal form property holds in this setting.

Lemma 6.15. $\ker \widetilde{\Psi}$ *is generated, as an* R*-module, by the syzygies*

$$s_{ij} := \frac{\mathsf{LT}(h_j)}{\gcd(\mathsf{LM}\, h_i, \mathsf{LM}\, h_j)} e_i - \frac{\mathsf{LT}(h_i)}{\gcd(\mathsf{LM}\, h_i, \mathsf{LM}\, h_j)} e_j. \qquad (i, j = 1 \ldots m) \tag{6.10}$$

Moreover,

$$\mathsf{LM}\, \langle s_{ij} \rangle = \langle \mathsf{LM}\, s_{ij} \rangle.$$

Proof: By Lemma 6.11, $\ker\widetilde{\Psi}$ is generated, as an $\mathbf{R}$-vector space, by binomials. Let $b := ax^\beta e_i - bx^\delta e_j$ be a binomial in $\ker\widetilde{\Psi}$. Then $ax^\beta \mathsf{LT}\, h_i = bx^\gamma \mathsf{LT}\, h_j$. This implies that b is a monomial multiple of s_{ij}; indeed, $b = (ax^\beta \gcd(\mathsf{LM}\, h_i, \mathsf{LM}\, h_j)/\mathsf{LT}\, h_j)s_{ij}$, proving the first claim. For the second claim, let $m \in \mathsf{LM}\ker\widetilde{\Psi}$, then $m = \mathsf{LM}\, b$ for some binomial $b \in \ker\widetilde{\Psi}$. By the same argument as before, m is a monomial multiple of some $\mathsf{LM}\, s_{ij}$. ∎

Now we can formulate the result of this section. Recall that $\{h_1, \ldots, h_k\}$ is called a Gröbner basis if $\mathsf{LM}\langle h_1, \ldots, h_k\rangle = \langle \mathsf{LM}\, h_1, \ldots, \mathsf{LM}\, h_k\rangle$. (For the definition of standard submodule basis, see Sect. 6.4.2.)

Theorem 6.16. *Let s_{ij} be generators (6.10) of $\ker\widetilde{\Psi}$ as an R-module, and let NF^Ψ be a normal form, and assume that*

$$\mathrm{NF}_r^\Psi(\Psi s_{ij}) = 0 \qquad \text{for all } s_{ij}.$$

Then:

a) $\{h_1, \ldots, h_k\}$ is a Gröbner basis for $\langle h_1, \ldots, h_k\rangle$.
b) $\{s_{ij} - \mathrm{NF}_\alpha^\Psi(\Psi s_{ij}) | 1 \le i < j \le k\}$ is a standard submodule basis for $\ker\Psi$.

Proof: Write $\alpha_{ij} = \mathrm{NF}_\alpha^\Psi(\Psi s_{ij})$, and define $u_{ij} := s_{ij} - \alpha_{ij}$. Let L be the free R-module generated by vectors v_{ij}, and define $\Phi : L \to M$ by $\Phi v_{ij} = u_{ij}$. Since $\mathrm{NF}_r^\Psi(\Psi s_{ij}) = 0$ we have $\Psi s_{ij} = \Psi\alpha_{ij}$, that is $\Psi u_{ij} = 0$. This shows that $\mathrm{Im}\,\Phi \subseteq \ker\Psi$.

The binomial s_{ij} lies in $\ker\widetilde{\Psi}$, and α_{ij} is a standard representation of Ψs_{ij}, hence by Lemma 6.12 it follows that $\mathsf{LM}\, s_{ij} > \mathsf{LM}\,\alpha_{ij}$, in other words $\mathsf{LM}\, s_{ij} = \mathsf{LM}\, u_{ij}$. Since $\langle s_{ij}\rangle_{1\le i<j\le k} = \ker\widetilde{\Psi}$, this shows that $\mathrm{Im}\,\widetilde{\Phi} = \langle \mathsf{LM}\, s_{ij}\rangle_{1\le i<j\le k} = \mathsf{LM}(\langle s_{ij}\rangle_{1\le i<j\le k}) = \mathsf{LM}\ker\widetilde{\Psi}$, the middle equality holding because of Lemma 6.15.

Finally, the map Φ has the normal form property, because the term orders involved are well-orders. The standard map theorem now applies. The statement $\mathrm{Im}\,\Phi = \ker\Psi$ means that the u_{ij} generate $\ker\Psi$. The statement that $\mathrm{Im}\,\widetilde{\Psi} = \mathsf{LM}\,\mathrm{Im}\,\Psi$ means that $\{h_1, \ldots, h_k\}$ is a Gröbner basis. Finally, $\mathrm{Im}\,\widetilde{\Phi} = \mathsf{LM}\,\mathrm{Im}\,\Phi$ implies that the u_{ij} form a standard submodule basis (relative to the induced order). This completes the proof. ∎

Buchberger's algorithm Most of the work has now been done. The final keystone is Buchberger's algorithm, which actually computes a Gröbner basis for arbitrary ideals.

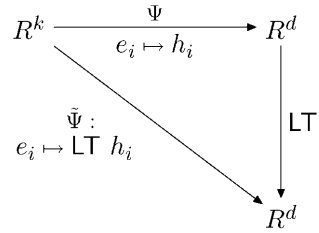

Fig. 6.4 Standard submodule basis diagram

Algorithm 6.17. *(Buchberger's algorithm)*
Input: $\{h_1, \ldots, h_p\} \subset \mathbf{R}[x]$
Output: *A Gröbner basis* $\{h_1, \ldots, h_k\} \subset \mathbf{R}[x]$ *for* $\langle h_1, \ldots, h_p \rangle$.
Algorithm:

$k \leftarrow p$
While $\mathrm{NF}_r^{\Psi}(s_{ij}) \neq 0$ *for any* $1 \leq i < j \leq k$, *do:*
  $h_{k+1} \leftarrow \mathrm{NF}_r^{\Psi}(s_{ij})$
  $k \leftarrow k + 1$
EndWhile

During execution, Ψ is supposed to be defined on $\oplus_{i=1}^{k} \mathbf{R}[x]e_i$, mapping e_i to h_i as usual, for the current value of k.
Proof of Algorithm 6.17: Since $s_{ij} \in \langle h_1, \ldots, h_k \rangle$ implies $\mathrm{NF}_r^{\Psi}(s_{ij}) \in \langle h_1, \ldots, h_k \rangle$, it follows by induction that $h_i \in \langle h_1, \ldots, h_n \rangle$ for any $i > n$. However the ideal $\langle \mathsf{LM}\, h_1, \ldots, \mathsf{LM}\, h_k \rangle$ does increase at each step, implying termination by Hilbert's basissatz. In turn, by Theorem 6.16 this implies that $\{h_1, \ldots, h_k\}$ is a Gröbner basis. ∎

6.4.2 Standard bases for submodules

A straightforward generalization of Gröbner bases gives a useful result for submodules. Gröbner bases are in fact a special case of standard submodule bases; see [GP88].

Let $\{h_1, \ldots, h_k\}$ be elements of $R^d := \oplus_{i=1}^{d} R\epsilon_i$. Here ϵ_i is the i-th basis vector of R^d, and R is the polynomial ring $\mathbf{R}[x_1, \ldots, x_n]$. Let M be the module $R^k := \oplus_{i=1}^{k} Re_i$, where the e_i denote the basis vectors of R^k. The R-module homomorphism $\Psi : M \to R^d$ is defined by $\Psi e_i = h_i$, so that $\mathrm{Im}\, \Psi = \langle h_1, \ldots, h_k \rangle_R$. See figure 6.4.

For monomials in R^d we take the elements of the form $x^{\alpha}\epsilon_i$, as usual, and $x^{\alpha}e_i$ in R^k. Also for the term order on R^d we take an arbitrary multiplicative well-order, and a refinement of it, via Ψ, on R^k. The generic algorithm 6.13 terminates for this setting, implementing a normal form map NF^{Ψ}.

Lemma 6.18. *Write* $\mathsf{LT}(h_i) = x^{\alpha_i} e_{n_i}$. *Then* $\ker \widetilde{\Psi}$ *is generated, as an* R*-module, by the syzygies*

$$s_{ij} = \frac{x^{\gamma_{ij}-\alpha_i}}{\mathsf{LC}\, h_i} e_{n_i} - \frac{x^{\gamma_{ij}-\alpha_j}}{\mathsf{LC}\, h_j} e_{n_j}, \tag{6.11}$$

where the indices i, j *run over the pairs with* $1 \le i < j \le k$ *and* $n_i = n_j$, *and* γ_{ij} *is the exponent of* $\operatorname{lcm}(x^{\alpha_i}, x^{\alpha_j})$. *Moreover,*

$$\mathsf{LM}\,\langle s_{ij}\rangle = \langle \mathsf{LM}\, s_{ij}\rangle,$$

where the indices i, j *run over the same values as in (6.11).*

Proof: The proof is a minor modification of the proof of Lemma 6.15. ∎

A basis $\{h_1, \ldots, h_k\}$ is called a *standard submodule-basis* for $\langle h_1, \ldots, h_k\rangle$ if $\langle \mathsf{LM}\, h_1, \ldots, \mathsf{LM}\, h_k\rangle = \mathsf{LM}\,\langle h_1, \ldots, h_k\rangle$. In terms of the map Ψ this means $\operatorname{Im}\widetilde{\Psi} = \mathsf{LM}\operatorname{Im}\Psi$. The result of this section is the following:

Theorem 6.19. *Let* s_{ij} *be the generators (6.11) of* $\ker\widetilde{\Psi}$ *as an* R*-module, and let* NF^{Ψ} *be a normal form, and assume that*

$$\mathrm{NF}^{\Psi}_r(\Psi s_{ij}) = 0 \qquad \text{for all } s_{ij}.$$

Then:

a) $\{h_1, \ldots, h_k\}$ *is a standard submodule basis for* $\langle h_1, \ldots, h_k\rangle$.
b) $\{s_{ij} - \mathrm{NF}^{\Psi}_{\alpha}(\Psi s_{ij})\}$, *where* ij *runs over the same pairs as in (6.11), is a standard submodule basis for* $\ker\Psi$.

Proof: The proof is word for word the same as that of Theorem 6.16, except that instead of invoking Lemma 6.15, one has to invoke Lemma 6.18. ∎

6.4.3 Standard bases for subalgebras

The bases we consider in this section are known as SAGBI bases (Subalgebra Analogue of Gröbner Bases for Ideals), see e.g. [Vas98]. Sturmfels calls them Canonical Subalgebra bases, see [Stu96, Ch. 11]. We call them 'standard subalgebra bases' to emphasize the similarity with the other cases.

The standard subalgebra basis criterion Let $\{g_1, \ldots, g_m\}$ be polynomials in $R = \mathbf{R}[x]$. Let $M := \mathbf{R}[y_1, \ldots, y_m]$ be the polynomial ring in m variables, and define the ring homomorphism $\Psi : y^{\alpha} \mapsto g_1^{\alpha_1} \cdots g_m^{\alpha_m}$. Then the image of Ψ is the subalgebra of R generated by $g_1, \ldots, g_m$, which we denote by $\mathbf{R}[g_1, \ldots, g_m]$.

In M and R we take the ordinary monomials, and as term order we take any multiplicative well-order, just as in the Gröbner basis case. With such a term order the monomial map Ψ is a ring homomorphism as well. Now we can define what a standard subalgebra basis is:

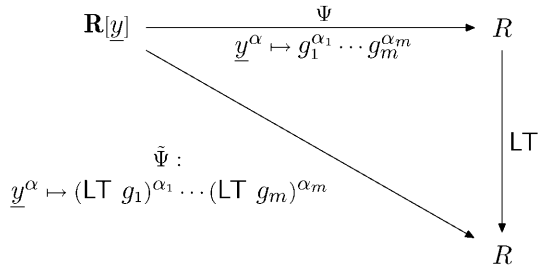

Fig. 6.5 Diagram for standard subalgebra bases

Definition 6.20. *The set* $\{g_1, \ldots, g_m\}$ *is called a* standard subalgebra basis *for the subalgebra* $\operatorname{Im} \Psi = \mathbf{R}[g_1, \ldots, g_m]$ *if* Ψ *is a standard map.*

Note that this is equivalent to: $\mathsf{LM}\, \mathbf{R}[g_1, \ldots, g_m] = \mathbf{R}[\mathsf{LM}\, g_1, \ldots, \mathsf{LM}\, g_m]$.

The criterion for standard subalgebra bases will be formulated in terms of a normal form map for Ψ. To apply this criterion we need an algorithm implementing the normal form. Algorithm6.13 implements such a normal form map NF^Ψ. Termination is guaranteed because $\mathsf{LM}\, r$ is strictly decreasing, and the term order is supposed to be a well-order. It is not yet a proper algorithm, since it is not explained how to decide whether a monomial is in $\operatorname{Im} \widetilde{\Psi}$ or not – but see below.

In order to prove the criterion in Theorem 6.23 below, we need two lemmas:

Lemma 6.21. *The kernel* $\ker \widetilde{\Psi}$ *is an ideal generated by binomials.*

Proof: $\ker \widetilde{\Psi}$ is an ideal since $\widetilde{\Psi}$ is a ring homomorphism, and it is generated as an $\mathbf{R}$-vector space by binomials by Lemma 6.11. A finite number of these binomials therefore generate $\ker \widetilde{\Psi}$ as an ideal. ∎

An element $\alpha \in M$ is called *homogeneous* if it is a sum of terms, each of which is mapped to the same monomial under $\widetilde{\Psi}$. By the previous lemma, $\ker \widetilde{\Psi}$ is generated by homogeneous elements. This notion is used in the following lemma, that is used to relax the condition on the generators of $\ker \widetilde{\Psi}$ in Theorem 6.23 below.

Lemma 6.22. *Let* $s_1, \ldots, s_p$ *be homogeneous generators of* $\ker \widetilde{\Psi}$ *as an ideal. Assume that* $\mathrm{NF}_r^\Psi(\Psi s_i) = 0$ *for all* $i = 1, \ldots, p$. *Then there exists a Gröbner basis* $\{s_i'\}_{i=1}^q$ *of* $\ker \widetilde{\Psi}$ *and representations* $\alpha_1', \ldots, \alpha_q'$ *with the properties (for all* $i = 1, \ldots, q$*):*

$$\left\{ \begin{array}{l} \text{All } s_i' \text{ are homogeneous,} \\ s_i' - \alpha_i' \in \langle s_j - \mathrm{NF}_\alpha^\Psi(\Psi s_j) \rangle_{j=1}^p, \\ \Psi \alpha_i' = \Psi s_i', \\ \widetilde{\Psi}\, \mathsf{LM}\, \alpha_i < \widetilde{\Psi}\, \mathsf{LM}\, s_i'. \end{array} \right. \tag{6.12}$$

Proof: The proof is by induction. First set $s'_i := s_i$ and $\alpha'_i := \mathrm{NF}^{\Psi}_{\alpha}(\Psi s_i)$, then properties (6.12) are satisfied. We turn $\{s'_i\}$ into a Gröbner basis by adding elements that increase $\langle \mathsf{LM}\, s'_i\rangle$, but leave (6.12) invariant.

Assume $\{s'_1, \ldots, s'_q\}$ is not yet a Gröbner basis. Choose an $s \in \ker \widetilde{\Psi}$ such that $\mathsf{LM}\, s \notin \langle \mathsf{LM}\, s'_1, \ldots, \mathsf{LM}\, s'_q\rangle$. Let $m := \widetilde{\Psi}\, \mathsf{LM}\, s$ and write $s = s_{=m} + s_{<m}$ where $s_{=m}$ is the homogeneous leading part of s, and $\widetilde{\Psi}\, \mathsf{LM}\, s_{<m} < m$. Now $\widetilde{\Psi} s_{=m}$ is just m times the coefficient of m in $\widetilde{\Psi} s$, which is zero, so $s_{=m} \in \ker \widetilde{\Psi}$ as well. Since $\ker \widetilde{\Psi} = \langle s'_1, \ldots, s'_q\rangle$ we can write

$$s_{=m} = \sum_{i=1}^{q} a_i s'_i. \qquad (a_i \in R)$$

Since $s_{=m}$ and the s'_i are homogeneous, we may assume that the a_i are homogeneous too. Now

$$\begin{aligned}
\widetilde{\Psi}\, \mathsf{LM}\, s_{=m} &= && \text{(definition)}\\
m &= && \text{(homogeneity)}\\
\widetilde{\Psi} \max_i \mathsf{LM}(a_i s'_i) &= && \text{(compatible term orders)}\\
\max_i \mathsf{LM}(a_i) \widetilde{\Psi}\, \mathsf{LM}\, s'_i &> && \text{(hypothesis)}
\end{aligned}$$

$$\begin{aligned}
\max_i \mathsf{LM}(a_i) \widetilde{\Psi}\, \mathsf{LM}\, \alpha'_i &= && \text{(compatible term orders)}\\
\max_i \widetilde{\Psi}\, \mathsf{LM}(a_i s'_i) &\geq && \text{(property of } \mathsf{LM})\\
\widetilde{\Psi}\, \mathsf{LM} \left(\sum_i a_i s'_i \right). &&&
\end{aligned}$$

So adding $s'_{q+1} := \sum_{i=1}^{q} a_i s'_i$ and $\alpha'_{q+1} := \sum_{i=1}^{q} a_i \alpha'_i$ to the generators and representations, leaves (6.12) invariant while $\langle \mathsf{LM}\, s'_i\rangle_{i=1}^{q+1} \supsetneq \langle \mathsf{LM}\, s'_i\rangle_{i=1}^{q}$. This proves the induction step. Since the ideal $\langle \mathsf{LM}\, s'_i\rangle_{i=1}^{q}$ cannot increase indefinitely, after a finite number of steps $\{s'_1, \ldots, s'_q\}$ is a Gröbner basis. ∎

The following theorem gives a criterion for standard subalgebra bases. It is the analogue of Theorem 6.16, which gives a criterion to recognize Gröbner bases: the S-polynomials should reduce to zero. In this case, the part of the S-polynomials is played by the binomial generators of the so-called toric ideal $\ker \widetilde{\Psi}$ (see e.g. [Stu96]).

Theorem 6.23. *Let $\{s_i\}_{i=1}^{p}$ be generators of $\ker \widetilde{\Psi}$ as an ideal, let NF^{Ψ} be a normal form, and assume that*

$$\mathrm{NF}^{\Psi}_{r}(\Psi s_i) = 0 \qquad \text{for } i = 1, \ldots, p.$$

Then:

a) $\{g_1, \ldots, g_m\}$ *is a standard basis for the subalgebra* $\mathbf{R}[g_1, \ldots, g_m]$.
b) $\{s_i - \mathrm{NF}_\alpha^\Psi(\Psi s_i)\}_{i=1}^p$ *generates* $\ker \Psi$.

Moreover, if $\{s_i\}_{i=1}^p$ *is a Gröbner basis for* $\ker \widetilde{\Psi}$, *then* $\{s_i - \mathrm{NF}_\alpha^\Psi(\Psi s_i)\}_{i=1}^p$ *is a Gröbner basis for* $\ker \Psi$.

Proof: If $\{s_i\}_{i=1}^p$ is a Gröbner basis, set $q = p$ and define $u_i := s_i - \mathrm{NF}_\alpha^\Psi(\Psi s_i)$, $i = 1, \ldots, q$. Since $s_i \in \ker \widetilde{\Psi}$, and $\mathrm{NF}_\alpha^\Psi(\Psi s_i)$ is a standard representation, it follows by Lemma 6.12 that $\mathsf{LM}\, u_i = \mathsf{LM}\, s_i$, and thus $\langle \mathsf{LM}\, u_i \rangle_{i=1}^q = \langle \mathsf{LM}\, s_i \rangle_{i=1}^q = \mathsf{LM} \langle s_i \rangle_{i=1}^q = \mathsf{LM} \ker \widetilde{\Psi}$. Otherwise use Lemma 6.22, and define $u_i := s_i' - \alpha_i'$, $i = 1, \ldots, q$. Then also $\mathsf{LM}\, u_i = \mathsf{LM}\, s_i'$, and $\langle \mathsf{LM}\, u_i \rangle_{i=1}^q = \langle \mathsf{LM}\, s_i' \rangle_{i=1}^q = \mathsf{LM} \langle s_i' \rangle_{i=1}^q = \mathsf{LM} \ker \widetilde{\Psi}$. Note that in both cases we have $u_i \in \langle s_j - \mathrm{NF}_\alpha^\Psi(\Psi s_j) \rangle_{j=1}^p$, for $i = 1, \ldots, q$.

Let L be the free M-module generated by the vectors $e_1, \ldots, e_q$, and define the M-module homomorphism Φ by $\Phi e_i := u_i$. By construction we have $\operatorname{Im} \Phi \subseteq \ker \Psi$, and by the foregoing discussion $\operatorname{Im} \widetilde{\Phi} = \langle \mathsf{LM}\, u_i \rangle_{i=1}^q = \mathsf{LM} \ker \widetilde{\Psi}$.

The map Φ has the normal form property – indeed, algorithm 6.2 provides a normal form – so we may apply the standard map theorem. The first conclusion, $\operatorname{Im} \Phi = \ker \Psi$, proves that $\{u_1, \ldots, u_q\}$ and hence $\{s_i - \mathrm{NF}_\alpha^\Psi(\Psi s_i)\}_{i=1}^p$ generates $\ker \Psi$. The conclusion $\operatorname{Im} \widetilde{\Phi} = \mathsf{LM} \operatorname{Im} \Phi$ implies that the $\{u_1, \ldots, u_q\}$ is a Gröbner basis for $\ker \Psi$. Finally, from $\operatorname{Im} \widetilde{\Psi} = \mathsf{LM} \operatorname{Im} \Psi$ we conclude that $\{g_1, \ldots, g_m\}$ is a standard subalgebra basis for $\mathbf{R}[g_1, \ldots, g_m]$. ∎

Implementing the criterion In order to check the standard subalgebra basis criterion, it is necessary to compute NF^Ψ, and also to compute a (Gröbner) basis for the ideal $\ker \widetilde{\Psi}$. These two problems are solved by the following algorithm.

Algorithm 6.24. *(Gröbner basis and normal form for binomial ideals, or: Finding syzygies for subalgebra bases, and representations of algebra elements.)*
Input: *A monomial ring homomorphism* $\widetilde{\Psi} : \mathbf{R}[y] \to \mathbf{R}[x]$, *an element* $m \in \mathbf{R}[x]$.
Output: *Gröbner basis for* $\ker \widetilde{\Psi}$; *a monomial* t *with* $\widetilde{\Psi} t = m$ *if it exists.*
Algorithm:

Introduce an elimination term order with $\{y_i\} < \{x_j\}$.
Compute a Gröbner basis $\mathcal{G}$ *of* $\langle y_1 - \widetilde{\Psi} y_1, \ldots, y_m - \widetilde{\Psi} y_m \rangle_{\mathbf{R}[y,x]}$ *with respect to* $<$.
Output $\mathcal{G} \cap \mathbf{R}[y]$
Let $t \in \mathbf{R}[y, x]$ *be the normal form of* m *with respect to* $\mathcal{G}$.
If $t \in \mathbf{R}[y]$, *output* t, *otherwise output "*$m \notin \operatorname{Im} \widetilde{\Psi}$*".*

See [Stu96, Alg. 4.5] or [AL94, Th. 4.3.13] for a proof. For more efficient algorithms to compute $\mathcal{G}$, see [Stu96, Ch. 12].

Using this, algorithm 6.13 can be implemented. In general, to compute $\mathrm{NF}^\Psi(f)$ one needs several invocations of algorithm 6.24. The slow Gröbner basis computation for $\mathcal{G}$ is only required once, and the normal form algorithm to compute t is much faster.

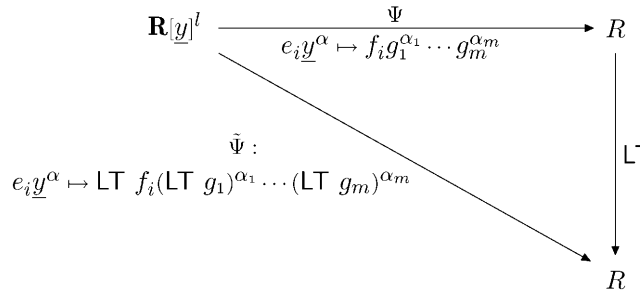

Fig. 6.6 Diagram for bases of modules over subalgebras

The analogue of Buchberger's algorithm If the condition "$\mathrm{NF}_r^{\Psi}(\Psi s_i) = 0$ for all s_i" fails, then adding the nonzero normal forms to the set of generators does not change the subalgebra, but does enlarge the set of monomials in the image of $\widetilde{\Psi}$. The condition can then be checked again, until it holds. This strategy is called the Buchberger algorithm in the case where $\mathrm{Im}\,\Psi$ is an ideal. In that case $\widetilde{\Psi}$ is also an ideal, and because of Hilbert's basissatz these ideals cannot increase indefinitely, assuring termination of the algorithm.

This argument fails in the case of subalgebras, and indeed, it can be shown that (finite) canonical subalgebra bases do not exist for all subalgebras. Perhaps the simplest example is the ideal $\langle x^2 \rangle \subset \mathbf{R}[x]$, considered as an algebra. A basis for this algebra includes polynomials P with $\mathsf{LM}\,P = x^p$, for every prime p. Another example is the ring of polynomials in x_1, x_2, x_3 that are invariant under cyclic permutation of the variables; see [Göb95] or [Stu96, Ch. 11] for details.

6.4.4 Standard bases for modules over subalgebras

As a generalization of the previous section, we now consider modules over subalgebras of polynomial rings. Standard bases for such objects are used in the sequel as building blocks for standard bases for left-right tangent spaces, the ultimate object of our interest.

The standard subalgebra basis criterion Let $\{g_1, \ldots, g_m\}$ and $\{f_1, \ldots, f_l\}$ be polynomials in $R = \mathbf{R}[x]$. The polynomial ring $\mathbf{R}[y_1, \ldots, y_m]$ is again denoted by M. Define the M-module homomorphism $\Psi : M^l \to R : e_i y^\alpha \mapsto f_i g_1^{\alpha_1} \cdots g_m^{\alpha_m}$. See Fig. 6.6; in this figure, e_i is the i-th canonical basis vector of $\mathbf{R}[\underline{y}]^l$. The image of Ψ, which is written as $\{f_1, \ldots, f_l\}\mathbf{R}[g_1, \ldots, g_m]$, is the object of interest. Algorithm 6.13 again implements a normal form. How to compute inverses of $\widetilde{\Psi}$ will be explained below.

The kernel of $\widetilde{\Psi}$ is an M-submodule, generated by binomials. Lemma 6.22 has an obvious counterpart in the current context, with "ideal" replaced by "M-submodule", and "Gröbner basis" by "standard submodule basis". The proof remains valid too.

The following theorem gives a criterion that guarantees Ψ to be a standard map, which, as usual, means that $\mathsf{LM}\,\mathrm{Im}\,\Psi = \mathrm{Im}\,\widetilde{\Psi}$. Here Ψ is defined in terms of two sets of elements, $\{f_i\}$ and $\{g_i\}$. The *pair* $(\{f_i\},\{g_i\})$ is called a *standard basis* for the subalgebra-module $\mathrm{Im}\,\Psi = \{f_i\}\mathbf{R}[g_i]$ if Ψ is a *standard map*. A basis is called a standard subalgebra-module basis if the following criterion is met:

Theorem 6.25. *Let* $\{s_i\}$ *be generators of* $\ker\widetilde{\Psi}$ *as an* M*-submodule, let* NF^{Ψ} *be a normal form, and assume that*

$$\mathrm{NF}^{\Psi}_r(\Psi s_i) = 0 \qquad \textit{for all } s_i.$$

Then:

a) $(\{f_i\},\{g_i\})$ *is a standard basis for the subalgebra-module*

$$\{f_1,\ldots,f_l\}\mathbf{R}[g_1,\ldots,g_m].$$

b) $\{s_i - \mathrm{NF}^{\Psi}_{\alpha}(\Psi s_i)\}$ *generates* $\ker\Psi$.

Moreover, if $\{s_i\}$ *is a standard submodule basis for* $\ker\widetilde{\Psi}$*, then* $\{s_i - \mathrm{NF}^{\Psi}_{\alpha}(\Psi s_i)\}$ *is a standard submodule basis for* $\ker\Psi$.

Proof: The proof is completely analogous to the proof of Theorem 6.23. ∎

Implementing the criterion To turn the above discussion into a computer program, we need an algorithm that compute generators s_i of $\ker\widetilde{\Psi}$ (that is, generators for the syzygies), and an algorithm implementing the normal form map NF^{Ψ}. In the application, the first module generator f_1 is 1, and this fact can be exploited. We specialize to this case. To describe the algorithm we introduce the following notation:

$$\begin{aligned} R_N &= \mathbf{R}[t_2,\ldots,t_l,y_1,\ldots,y_m,x_1,\ldots,x_n],\\ I_N &= \langle t_2 - \mathsf{LM}\,f_2,\ldots,t_l - \mathsf{LM}\,f_l, y_1 - \mathsf{LM}\,g_1,\ldots,y_m - \mathsf{LM}\,g_m\rangle,\\ G_N &= \{g_{N1},\ldots,g_{Nq}\} = \text{Gröbner basis for } I_N \text{ with respect to } \preccurlyeq, \end{aligned}$$

where $\preccurlyeq$ is an elimination term order on R_N with $\{y_i\} \preccurlyeq \{t_i\} \preccurlyeq \{x_i\}$, and which is *graded*, with respect to the total degree, in the variables t_i. Then we have:

Proposition 6.26. *With the definitions above, each binomial in*

$$G_N \cap \{1,t_2,\ldots,t_l\}\mathbf{R}[y_1,\ldots,y_m]$$

is an element of $\ker\widetilde{\Psi}$ *via the binomial correspondence*

$$\begin{aligned} mt_i - m't_j &\mapsto me_i - m'e_j,\\ mt_i - m' &\mapsto me_i - m'e_1,\\ m - m' &\mapsto me_k - m'e_k \quad \textit{for } k = 1,\ldots,l \end{aligned}$$

Here m *and* m' *are arbitrary monomials in* $\mathbf{R}[y]$*. These elements together form a generating set of* $\ker\widetilde{\Psi}$ *as an* $\mathbf{R}[y]$*-module.*

Proof: First note that $I_N \cap \{1, t_2, \ldots, t_l\}\mathbf{R}[y] \subseteq \ker \widetilde{\Psi}$ via the binomial correspondence. For the converse, of the binomials in $\ker \widetilde{\Psi}$ not generated by the elements mentioned above, let α be one with smallest leading monomial. This leading monomial is of the form $t_i m$ or m, with $m \in \mathbf{R}[y]$. Since G_N is a Gröbner basis of I_N, this monomial is a multiple of the leading monomial of some $g \in G_N$. Both terms of g are at most linear in the t_i, and do not involve x_i, by the choice of term order. Subtracting the proper multiple of g from α yields an α' with smaller leading monomial, which by choice of term order again, lies in $G_N \cap \{1, t_2, \ldots, t_l\}$. This provides the required contradiction. ∎

Remark 6.27. *(Efficiency)* It is most efficient to first enlarge the $\{g_i\}$ to a standard subalgebra basis. Then elements $m - m' \in G_N$, each resulting in l syzygies to be checked in the condition of Theorem 6.25, can all be ignored since they will automatically reduce to 0.

Using the Gröbner basis G_N, and the standard map Ψ_N related to it, we can write down a normal form algorithm for Ψ. It is algorithm 6.13 with an explicit subroutine for finding the inverse image of a monomial under $\widetilde{\Psi}$. Here the term order $\preccurlyeq$ is essential.

Algorithm 6.28. *(Normal form for modules over subalgebras)*
Input: *A map* $\Psi : \mathbf{R}[y]^l \to R$ *with* $\Psi e_1 = 1$, *the associated monomial map* $\widetilde{\Psi}$, *a Gröbner basis* G^N *as above, the associated map* $\Psi_N : R_N^q \to R_N$, *and the associated normal form map* NF^{Ψ_N}.
Output: $\alpha \in \mathbf{R}[y]^l$, $r \in R$ *such that*

1. $f = r + \Psi\alpha$,
2. $r = 0$ *or* $\mathsf{LM}\, r \notin \operatorname{Im} \widetilde{\Psi}$,
3. α *is a standard representation.*

Algorithm:

$\alpha \leftarrow 0$
$r \leftarrow f$
BeginLoop
  $m \leftarrow (\mathsf{LT}\, r)$ *with* e_i *replaced by* t_i.
  $r_N \leftarrow \mathrm{NF}_r^{\Psi_N}(m)$
  If $r_N \in \{1, t_2, \ldots, t_l\}\mathbf{R}[y]$, *then*
    If $r_N \in \mathbf{R}[y]$, *then*
      $t \leftarrow e_1 r_N$
    Else
      $t \leftarrow r_N$ *with* t_i *replaced by* e_i
    EndIf
    $\alpha \leftarrow \alpha + t$
    $r \leftarrow r - \Psi t$

Else
ExitLoop
EndIf
EndLoop

Proof: The condition that $\mathsf{LT}\, r \in \operatorname{Im} \widetilde{\Psi}$ is equivalent to the existence of an $r_N \in \{1, t_2, \ldots, t_l\}\mathbf{R}[y]$ that is equal to m modulo I_N. Since the term order favors monomials without x_i's and with lowest-degree t_i's, if such a form exists it is the output of the normal form $\mathrm{NF}_r^{\Psi_N}(m)$. Correctness follows from the invariant $f = r + \Psi\alpha$, and the fact that $\mathsf{LM}\, r$ decreases implies termination, using the well-orderedness of $\preccurlyeq$. ∎

The analogue of Buchberger's algorithm If the basis $(\{f_i\}, \{g_i\})$ defining the map Ψ, is *not* a standard basis, that is $\mathrm{NF}_r^{\Psi}(\Psi s_i) \neq 0$ for some s_i, it may be turned into one by adding elements. There are two possibilities: Either the binomial s_i is of the form $e_j(y^\alpha - y^\beta)$, or it is of the more general form $e_j y^\alpha - e_k y^\beta$ with $j \neq k$. Syzygies of the first form will not occur if $\{g_i\}$ is a standard subalgebra basis to start with (see remark 6.27). Using the normal form of syzygies of the second form, new elements are found and added to the $\{f_i\}$, that increase $\operatorname{Im} \widetilde{\Psi}$ but leave $\operatorname{Im} \Psi$ invariant.

6.4.5 Left-Right tangent space

In Sect. 3.2.3 the codimension of the tangent space to the orbit of a map under left-right transformations was shown to be equal to the codimension of

$$T^r_{\boldsymbol{E}} \;=\; J + \{1, f_1, f_2\}\mathbf{R}[[H, H_2]] \subset R. \tag{6.13}$$

Here R is the ring of formal power series $\mathbf{R}[[\rho_1, \rho_2, \psi, \chi]]$, J is an ideal, and f_1, f_2, H, H_2 are all elements of R. The first problem to solve is: How to compute the codimension of T, and find elements in R complementing it. The second problem is: Given an arbitrary $f \in R$, write it explicitly as a sum of an element of T and a linear combination of the complementing elements. The result of this latter procedure can be used to build explicit reparametrizations connecting an arbitrary deformation to a universal one.

The situation is analogous to the case of unfoldings of functions under right-transformations. There the tangent space was an ideal, the procedure to find the codimension and complementing elements was Buchberger's algorithm (for standard bases), and the normal form procedure computed the required representation for arbitrary functions f, which the algorithm of Kas and Schlessinger used to compute reparametrizations.

The space of equation (6.13) is of the following general form:

$$\langle h_1, \ldots, h_k \rangle + \{f_1, \ldots, f_l\}\mathbf{R}[[g_1, \ldots, g_m]]. \tag{6.14}$$

Spaces of this form are the image of Ψ in Fig. 6.7, except that for simplicity we here use the polynomial ring $R = \mathbf{R}[x]$ as a base ring, instead of the ring of formal

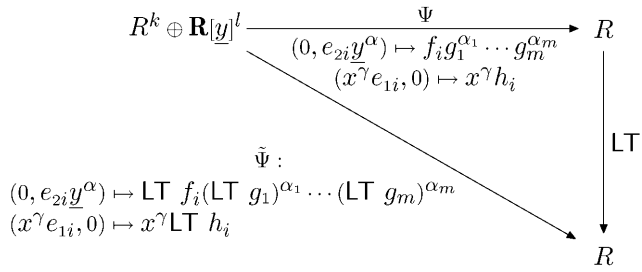

Fig. 6.7 LR-tangent space diagram

power series $\mathbf{R}[[x]]$. In the application we use truncated formal power series, but at this stage this would unnecessarily complicate the notation. Note that the discussion below is independent of the term order; indeed, in the polynomial ring the term order may be, but need not be, a well-order. (However, for a reversed well-order, the reduced normal form does not exist; algorithm 6.14 will not terminate in general.)

In Fig. 6.7, the maps Ψ and $\widetilde{\Psi}$ are defined on elements of the form $(x^\alpha e_{1i}, 0)$ and $(0, e_{2i}y^\alpha)$. By linearity, Ψ is defined on all of on all of $R^k \oplus \mathbf{R}[\underline{y}]^l$. The goal of this section is, first of all, to formulate a condition guaranteeing that $\mathsf{LM}\,\mathrm{Im}\,\Psi = \mathrm{Im}\,\widetilde{\Psi}$. If equality holds, the map Ψ is called a *standard map*, and the triple

$$(\{h_1, \ldots, h_k\}, \{f_1, \ldots, f_l\}, \{g_1, \ldots, g_m\})$$

is called a *standard basis* for the "left-right tangent space" (6.14).

Generators of ker $\widetilde{\Psi}$ In what follows, two natural but different $\mathbf{R}[\underline{y}]$-module structures on $R^k \oplus \mathbf{R}[\underline{y}]^l$ will be used. For lack of better names, we call them the (ordinary) $\mathbf{R}[\underline{y}]$- and monomial $\mathbf{R}[\underline{y}]$-module structures, and multiplication is defined as follows:

$$\begin{aligned} y_i \cdot (a, b) &= (g_i a, y_i b), && \text{(ordinary module structure)} \\ y_i \cdot (a, b) &= (\mathsf{LM}(g_i)a, y_i b). && \text{(monomial module structure)} \end{aligned}$$

By Lemma 6.11 the kernel $\ker\widetilde{\Psi}$ is generated, as a linear vector space, by binomials. Write $\widetilde{\Psi}_1$ for the restriction of $\widetilde{\Psi}$ to $R^k \times \{0\}$, whose image is the ideal $\langle h_i \rangle$, and let $\widetilde{\Psi}_2$ be the restriction of $\widetilde{\Psi}$ to $\{0\} \times \mathbf{R}[\underline{y}]^l$. Then we have the following:

Lemma 6.29. *The kernel of $\widetilde{\Psi}$ is generated, as a linear vector space, by the following three sets of binomials:*

a) $\{(x^\alpha e_{1i} - cx^\beta e_{1j}\ ,\ 0) \in \ker\widetilde{\Psi}_1\}_{1\le i\le j\le k}$
b) $\{(0\ ,\ y^\alpha e_{2i} - cy^\beta e_{2j}) \in \ker\widetilde{\Psi}_2\}_{1\le i\le j\le l}$
c) $\{(x^\alpha e_{1i}\ ,\ -cy^\beta e_{2j}) \ \mid\ \widetilde{\Psi}_1(x^\alpha e_{1i}) = \widetilde{\Psi}_2(cy^\beta e_{2j})\}_{1\le i\le k,\ 1\le j\le l}$

Moreover, any binomial in the kernel is in one of these three sets.

Proof: By Lemma 6.11, $\ker\widetilde{\Psi}$ is generated by binomials. A binomial in $R^k \oplus \mathbf{R}[\underline{y}]^l$ has both monomials in the first component, both in the second, or exactly one in either, corresponding to the three cases of the lemma. ∎

The binomials of case (a) generate (as vector space) an R-module. It is generated (as an R-module) by the ideal syzygies given in Lemma 6.15. The binomials of case (b) generate (as a vector space) a monomial $\mathbf{R}[y]$-module. Generators of this object are just the subalgebra-module syzygies, which can be computed by algorithm 6.28. It remains to find the binomials of case (c). The linear span of these binomials has a monomial $\mathbf{R}[\underline{y}]$-module structure. The projection of this module onto its second component, $\mathbf{R}[\underline{y}]^l$, is the monomial $\mathbf{R}[\underline{y}]$-module $\widetilde{\Psi}_2^{-1}(\mathsf{LM}\,\langle h_1,\ldots,h_k\rangle)$. This module is generated by monomials. Assuming for the moment that we know its monomial generators $v_i \in \mathbf{R}[\underline{y}]^l$, we can say the following:

Lemma 6.30. *Let $\{s_1,\ldots,s_p\} \subset R^k$ be the ideal syzygies of $\langle h_i\rangle$ as defined in lemma 6.15. Let $\{b_1,\ldots,b_q\} \subset \mathbf{R}[\underline{y}]^l$, be a binomial standard basis of the submodule $\ker\widetilde{\Psi}_2$ (the subalgebra-module syzygies). Let $v_1,\ldots,v_r$ be monomial generators of the submodule $\widetilde{\Psi}_2^{-1}(\mathsf{LM}\,\langle h_1,\ldots,h_k\rangle) \subseteq \mathbf{R}[\underline{y}]^l$. Let $<$ on $R^d \oplus \mathbf{R}[\underline{y}]$ be a refinement of the term-order on R, such that if $m \in \mathbf{R}[\underline{y}]$ and $m' \in R^d$ and $m - m' \in \ker\widetilde{\Psi}$, then $m > m'$. Then, for any $m \in \mathsf{LM}\ker\widetilde{\Psi}$, (at least) one of the following holds:*

a) $m \in \langle \mathsf{LM}\,s_1,\ldots,\mathsf{LM}\,s_p\rangle_R \times \{0\}$,
b) $m \in \{0\} \times \langle \mathsf{LM}\,b_1,\ldots,\mathsf{LM}\,b_q\rangle_{\mathbf{R}[\underline{y}]}$,
c) $m \in \{0\} \times \langle v_1,\ldots,v_r\rangle_{\mathbf{R}[\underline{y}]}$.

Proof: Let $m \in \mathsf{LM}\ker\widetilde{\Psi}$. Then there exists an $m' < m$ such that $m - m' \in \ker\widetilde{\Psi}$. This binomial falls into one of the classes (a), (b) or (c) of Lemma 6.29. In case (a), $m \in \mathsf{LM}\,\langle s_i\rangle \times \{0\} = \langle \mathsf{LM}\,s_i\rangle \times \{0\}$, by Lemma 6.18. In case (b), $m \in \{0\} \times \mathsf{LM}\,\langle b_i\rangle = \{0\} \times \langle \mathsf{LM}\,b_i\rangle$, since $\{b_i\}$ is a standard submodule basis by hypothesis. In case (c) finally, $m \in \{0\} \times \mathbf{R}[\underline{y}]^l$ by the choice of the refined term order. This implies $m' \in \mathsf{LM}\,\langle h_i\rangle \times \{0\}$, hence $m \in \widetilde{\Psi}_2^{-1}\,\mathsf{LM}\,\langle h_i\rangle = \{0\} \times \langle v_i\rangle$. ∎

Now we can describe $\ker\widetilde{\Psi}$ completely:

Lemma 6.31. *Let $\{s_1,\ldots,s_p\}$, $\{b_1,\ldots,b_q\}$ and $\{v_1,\ldots,v_r\}$ be as in Lemma 6.30. Let $w_1,\ldots,w_r \in R^k$ be monomials such that $v_i - w_i \in \ker\widetilde{\Psi}$ for all $1 \le i \le r$. (Here $v_i - w_i$ denotes the element $(-w_i, v_i) \in R^d \oplus \mathbf{R}[\underline{y}]^l$.) Then*

$$\begin{aligned}\ker\widetilde{\Psi} = {} & \langle s_1,\ldots,s_p\rangle_R \times \{0\} \\ & + \{0\} \times \langle b_1,\ldots,b_q\rangle_{\mathbf{R}[\underline{y}]} \\ & + \langle v_1 - w_1,\ldots,v_r - w_r\rangle_{\mathbf{R}[\underline{y}]}\end{aligned} \tag{6.15}$$

where in the last term the monomial $\mathbf{R}[\underline{y}]$-module structure is used.

Proof: By Lemma 6.11 it is enough to check that every *binomial* in $\ker\widetilde{\Psi}$ lies in the right-hand-side of (6.15). So let $m - m'$ be any binomial in $\ker\widetilde{\Psi}$, and suppose $m > m'$. If $m - m'$ is an element of the sets (a) or (b) of Lemma 6.29, then it is an element of $\langle s_i\rangle_R \times \{0\}$ or $\{0\} \times \langle b_i\rangle_{\mathbf{R}[\underline{y}]}$ respectively, since $\{s_i\}$ and $\{b_i\}$ generate the corresponding submodules. If $m-m'$ is in set (c), then by Lemma 6.30(c), $m \in \{0\}\times\langle v_i\rangle_{\mathbf{R}[\underline{y}]}$, say $m = y^\alpha v_i$. Then $(m-m')-y^\alpha(v_i-w_i)$ is in set (b), and these elements are in the right-hand-side of (6.15) as was already shown. ∎

Standard bases for left-right tangent spaces In order to formulate the theorem, we now introduce the map Φ. As in previous cases, the image of Φ will be the kernel of Ψ, if suitable conditions are met. Recall that Ψ_1 is the restriction of Ψ to $R^k \times \{0\}$.

Definition 6.32. *Let $\{s_1,\dots,s_p\}$, $\{b_1,\dots,b_q\}$, $\{v_1-w_1,\dots,v_r-w_r\}$ be as in Lemma 6.31. Let ϵ_{1i} be canonical basis vectors of R^p, let ϵ_{2i} be those of $\mathbf{R}[\underline{y}]^q$ and ϵ_{3i} those of $\mathbf{R}[\underline{y}]^r$. Suppose both Ψ and Ψ_1 have the normal form property, and let NF^{Ψ}, NF^{Ψ_1} be normal forms. Then Φ is defined as follows:*

$$\begin{aligned}
\Phi : R^p \oplus \mathbf{R}[\underline{y}]^q \oplus \mathbf{R}[\underline{y}]^r \to\ & R^k \oplus \mathbf{R}[\underline{y}]^l \\
x^\alpha\epsilon_{1i} \mapsto\ & (x^\alpha s_i, 0) - \left(x^\alpha\,\mathrm{NF}^{\Psi_1}_\alpha(\Psi_1 s_i), 0\right) \quad (i = 1\dots p)\\
y^\beta\epsilon_{2i} \mapsto\ & \left(0, y^\beta b_i\right) - y^\beta\,\mathrm{NF}^{\Psi}_\alpha(\Psi(0,b_i)) \quad (i = 1\dots q)\\
y^\beta\epsilon_{3i} \mapsto\ & y^\beta(v_i-w_i) - y^\beta\,\mathrm{NF}^{\Psi}_\alpha(\Psi(v_i-w_i)) \quad (i = 1\dots r)
\end{aligned}$$

Wherever appropriate, multiplication is according to the ordinary $\mathbf{R}[\underline{y}]$*-module structure.*

The definition of Φ on $R^p \times \{0\} \times \{0\}$ uses the normal form $\mathrm{NF}^{\Psi_1}_\alpha$ instead of $\mathrm{NF}^{\Psi}_\alpha$, since the latter is not guaranteed to map into $R^k \times \{0\}$, and only on that subset can we use the R-module structure, instead of the coarser structure of an $\mathbf{R}[\underline{y}]$-module.

Extending the ordinary $\mathbf{R}[\underline{y}]$-module structure in the obvious way to $R^p \oplus \mathbf{R}[\underline{y}]^q \oplus \mathbf{R}[\underline{y}]^r$, the map Φ becomes an ordinary $\mathbf{R}[\underline{y}]$-module homomorphism. The map $\widetilde{\Psi}$ is defined as usual, by setting $\widetilde{\Psi}m := \mathsf{LM}\,\Psi m$ for monomials, and extending it linearly. Then, similarly extending the *monomial* $\mathbf{R}[\underline{y}]$-module structure to $R^p \oplus \mathbf{R}[\underline{y}]^q \oplus \mathbf{R}[\underline{y}]^r$, makes $\widetilde{\Phi}$ a monomial $\mathbf{R}[\underline{y}]$-module homomorphism.

Theorem 6.33. *Let $\{s_i\} \subset R^k$, $\{b_i\} \subset \mathbf{R}[\underline{y}]^l$ and $\{v_i - w_i\} \subset R^k \oplus \mathbf{R}[\underline{y}]^l$ be as in Lemma 6.31. Let Φ be defined as above. Suppose that*

$$\begin{aligned}
&1.\ \mathrm{NF}^{\Psi_1}_r(\Psi_1 s_i) = 0, && (i = 1\dots p)\\
&2.\ \mathrm{NF}^{\Psi}_r(\Psi(0,b_i)) = 0, && (i = 1\dots q)\\
&3.\ \mathrm{NF}^{\Psi}_r(\Psi(v_i-w_i)) = 0. && (i = 1\dots r)
\end{aligned}$$

Then:

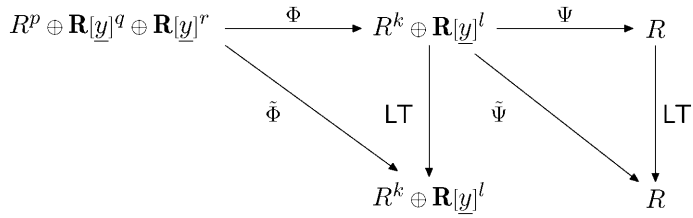

Fig. 6.8 The full LR-tangent space diagram

a) The triple

$$(\{h_1, \ldots, h_k\}, \{f_1, \ldots, f_l\}, \{g_1, \ldots, g_m\})$$

forms a standard basis for

$$\operatorname{Im} \Psi = \langle h_1, \ldots, h_k \rangle_R + \{f_1, \ldots, f_l\}\mathbf{R}[g_1, \ldots, g_m].$$

b) The map Φ is a standard map, and $\operatorname{Im} \Phi = \ker \Psi$.

Proof: Assumptions 1 to 3, together with the definition of Φ as an ordinary $\mathbf{R}[\underline{y}]$-module homomorphism, imply that $\operatorname{Im} \Phi \subseteq \ker \Psi$. By Lemma 6.12 applied to $\bar{\Psi}$ and Ψ_1 and their corresponding normal forms, we find the following:

$$\begin{aligned}
\mathsf{LM}\, \Phi \underline{x}^\alpha \epsilon_{1i} &= \mathsf{LM}(\underline{x}^\alpha s_i, 0) \\
\mathsf{LM}\, \Phi \underline{y}^\beta \epsilon_{2i} &= \mathsf{LM}(0, \underline{y}^\beta b_i) \\
\mathsf{LM}\, \Phi \underline{y}^\beta \epsilon_{3i} &= \mathsf{LM}\, \underline{y}^\beta (v_i - w_i) = \mathsf{LM}\, \underline{y}^\beta v_i
\end{aligned}$$

Using Lemma 6.31 it follows that $\operatorname{Im} \widetilde{\Phi} \supseteq \mathsf{LM} \ker \widetilde{\Psi}$.

The map Φ has the normal form property, because the term orders involved are well-orders. The standard map theorem 6.10 applies, from which the results $\operatorname{Im} \Phi = \ker \Psi$ and $\operatorname{Im} \widetilde{\Phi} = \mathsf{LM} \operatorname{Im} \Phi$ follow immediately. The statement $\operatorname{Im} \widetilde{\Psi} = \mathsf{LM} \operatorname{Im} \Psi$ means that the triple $(\{h_i\}, \{f_i\}, \{g_i\})$ forms a standard basis for $\operatorname{Im} \Psi$. This completes the proof. ∎

Implementing the criterion The map $\mathrm{NF}_r^{\Psi_1}$ of Theorem 6.33 is just the normal form algorithm 6.2 for ideals. The algorithm that computes NF_r^{Ψ} is the 'union' of algorithm 6.28 for modules over subalgebras, and algorithm 6.2: For each leading term, try to write it as an ideal element and, when that fails, try to write it as an element of the subalgebra-module.

In sections 6.4.1 and 6.4.4 it is explained how to compute syzygies of the first and second kind, the s_i and b_i of Theorem 6.33. We now assume that $\{h_1, \ldots, h_k\}$ is already a standard ideal basis. This means that the $\Psi_1 s_i$ reduce to zero so that we can forget about syzygies of the first kind. It remains to find generators v_i of the $\mathbf{R}[y]$-module $\widetilde{\Psi}_2^{-1}(\mathsf{LM} \langle h_1, \ldots, h_k \rangle)$. Using the notation of section 6.4.4, this submodule is generated by the monomials in $\langle I_N \cup \{\mathsf{LM}\, h_1, \ldots, \mathsf{LM}\, h_k\} \rangle$ intersected with $\{1, t_2, \ldots, t_l\}\mathbf{R}[y]$ (here we use that $\{h_1, \ldots, h_k\}$ is a standard

basis). A possible line of attack is therefore: Compute generators of the largest monomial ideal contained in $\langle I_N \cup \{\mathsf{LM}\, h_1, \ldots, \mathsf{LM}\, h_k\}\rangle \cap \mathbf{R}[t, y]$ (which is automatically a Gröbner basis), and select the elements in $\{1, t_2, \ldots, t_l\}\mathbf{R}[y]$. With the proper term order, this amounts to one Gröbner basis calculation, and one 'inverse' Gröbner basis calculation. More precisely, this is the algorithm:

Algorithm 6.34. *(Computing syzygies $v_i - w_i$ of the third kind)*
Input: *A basis G for the ideal I_N (see Prop. 6.26), a term order $\preccurlyeq$ with $\{y_i\} \preccurlyeq \{t_i\} \preccurlyeq \{x_i\}$ which is graded in the t_i, a map Ψ with $\Psi e_{21} = f_1 = 1$, and such that $\{\Psi e_{11}, \ldots, \Psi e_{1k}\} = \{h_1, \ldots, h_k\}$ is a Gröbner basis.*
Output: *Generators $v_1, \ldots, v_r$ of the monomial ideal $\widetilde{\Psi}_2^{-1}(\langle \mathsf{LM}\, h_1, \ldots, \mathsf{LM}\, h_k\rangle)$, corresponding elements $w_1, \ldots, w_r \in R^k$ with $\widetilde{\Psi}_2 v_i = \widetilde{\Psi}_1 w_i$.*
Algorithm:

Compute Gröbner basis G' for $\langle G \cup \{\mathsf{LM}\, h_1, \ldots, \mathsf{LM}\, h_k\}\rangle$ with respect to $\preccurlyeq$.
Compute $G'' = G' \cap \{1, t_2, \ldots, t_l\}\mathbf{R}[y]$
Find basis M of largest monomial subideal contained in $\langle G''\rangle \subseteq \mathbf{R}[t, y]$
$M \leftarrow M \cap \{1, t_2, \ldots, t_l\}\mathbf{R}[y]$ and label the elements $v_1, v_2, \ldots, v_r$
For $i = 1, \ldots, r$, do the following:
    Find $m \in \mathbf{R}[x]$ and j such that $\widetilde{\Psi}_2 v_i = m\, \mathsf{LM}\, h_j$
    $w_i \leftarrow m e_{1j}$
EndFor

Proof: By the choice of term order $\preccurlyeq$, the elements in $B \cap \{1, t_2, \ldots, t_l\}\mathbf{R}[y]$ generate $\langle B\rangle \cap \{1, t_2, \ldots, t_l\}\mathbf{R}[y]$, for any Gröbner basis B, hence the output $\{v_1, \ldots, v_r\}$ is indeed a basis for the largest monomial ideal contained in $I_N \cap \{1, t_2, \ldots, t_l\}\mathbf{R}[y]$. The body of the final For-loop is simply the normal form algorithm NF^{Ψ_1} for ideals written down explicitly – it takes only one pass for monomials. ∎

Remark 6.35. *(A shortcut)* If the basis G is a Gröbner basis for I_N to start with, the computation of G' is a bit easier: Only syzygies of monomials and binomials need to be checked. The result is a monomial, and its reduction is either zero or a single new monomial.

To compute the basis M in line 3 of above algorithm, note that a monomial is in $\langle G''\rangle$ if and only if it is reduced to 0 by the ordinary normal form algorithm 6.2. The basis G'' consists of binomials and monomials only (because the S-polynomial that occur are either monomials or binomials; see [ES96, Prop. 1.1]). A reduction by a binomial results in a (nonzero) monomial, whereas a monomial reduction results in 0. Therefore, a basis for the largest monomial ideal contained in M can be found by running the normal form algorithm backwards via the binomials, starting from the monomials in G'':

Algorithm 6.36. *(Finding largest monomial subideal)*
Input*: A Gröbner basis G consisting of monomials and binomials only.*

Output: *A (Gröbner) basis M for largest monomial ideal $\langle M\rangle \subset \langle G\rangle$*
Algorithm:

$M \leftarrow$ all monomials of G
$B \leftarrow$ all binomials of G
For all monomials x^α in M, do the following:
  For all binomials $x^\beta - x^\gamma$ in B (where $x^\beta > x^\gamma$) do
    If $x^{\beta+(\alpha-\gamma)^+} \notin \langle M\rangle$, then
      Add it to M
    EndIf
  EndFor
EndFor
Output M

Here α^+, where α is a vector in $\mathbb{Z}^n$, denotes the vector α with all negative entries replaced by zeroes.

Proof: Let m be a monomial in $\langle G\rangle$. We shall prove: If m can be reduced via a $g \in G$ to m' and $m' \in \langle M\rangle$, then $m \in \langle M\rangle$. Since G is a Gröbner basis a finite number of reduction steps yield a monomial m'' which is a multiple of some monomial in G. Since M contains all monomials of G this means that that $m'' \in \langle M\rangle$, hence by 'backward induction' $m \in \langle M\rangle$.

Write $g = x^\beta - x^\gamma$ and $m = x^\delta$, and assume that $\mathsf{LM}\, g = x^\beta | m = x^\delta$, then the reduct of m by g is $x^{\delta-\beta+\gamma}$. Assume that this monomial is in $\langle M\rangle$, say $x^\alpha | x^{\delta-\beta+\gamma}$ with $x^\alpha \in M$. Using $x^\beta | x^\delta$ it follows that $x^{(\alpha-\gamma)^+} | x^{\delta-\beta}$, and after multiplication with x^β this becomes $x^{\beta+(\alpha-\gamma)^+} | x^\delta$. By the algorithm, $x^{\beta+(\alpha-\gamma)^+} \in \langle M\rangle$, hence $x^\delta \in \langle M\rangle$. ∎

Remark 6.37. *(Efficiency)* When a monomial m is added to M, it may render other monomials in M redundant, namely those that are multiples of m. The algorithm becomes a little more efficient if these are removed.

6.5 The ring of formal power series

The analogue of a Gröbner basis, in the ring of formal power series, is commonly called a *standard basis*.[3] Introduced in 1964 by Hironaka [Hir64], it preceded the notion of Gröbner basis, which was introduced by Buchberger in his thesis of 1965, see [Buc65]. Meanwhile it has been shown (see e.g. [Bec90a, Bec90b, Bec93]) that concepts in one context have close analogues in the other. The aim of this section is to show that the standard basis criteria developed above, for various subsets of the polynomial ring, have similar analogues in the ring of formal power series. Because of the modular (or 'object-oriented') set-up, we only have to show that the normal form property continues to hold.

[3] In this chapter we call such bases *standard ideal bases*, and use the term *standard basis* in the more general sense explained before.

6.5.1 Three approaches

A polynomial has finitely many terms, so whatever term order is chosen, the *leading* or highest term is well-defined. This is not so for formal power series. In that case, to guarantee existence of the leading term, the term order should be a reversed well-order, that is, every nonempty subset of terms should have a *highest* term. In this section we suppose that the term order is a reversed well-order.

Unless the number of monomials is finite, a reversed well-order is not a well-order, and the normal form algorithm 6.2 will not terminate in general, so existence of a normal form becomes an issue. There are different ways to approach this problem. Instead of looking at all ideals, one can restrict to ideals generated by *polynomials* (cf. [CLO98, p. 165]). It turns out that such ideals admit standard bases consisting of rational functions. Moreover a suitable normal form algorithm exists, acting within the ring of rational functions. This algorithm is known as the *Mora normal form*, see [Mor82, Mor85], and leads to the theory of standard ideal bases in local rings, see [GP88].

Another way of dealing with the infiniteness is to *truncate*. This makes the set of monomials finite, and hence recovers the well-ordering property of the term order. Although Mora's approach is more elegant than the method of truncation, the former only seems to work in the context of ideals. For our intended application we also need subalgebra bases, which are infinite in general, so that we need to truncate anyhow. See section 6.5.3 for more details.

A third approach is to let go of the algorithmic character altogether, and define the normal form map inductively. In this way we recover Hironaka's result that every ideal has a standard basis, see also [Bec93]. This is the subject of the next section.

6.5.2 Existence of a normal form for formal power series

In order to prove that a normal form map exists, we need some lemmas. We assume the term order is a reversed well-order.

Lemma 6.38. *Let $u \neq 1$ be a monomial, and assume u has no successor. Then there exists a strictly decreasing sequence of monomials u^i, $i \in \mathbb{N}$ such that $u = \inf_{i\in\mathbb{N}} u^i$.*

Proof: The set of monomials is countable. Let w^i be a counting of them, without duplicates, and assume $w^0 = 1$. Set $u^0 = 1$ and define

$$u^{i+1} := w^j \quad \text{with} \quad j := \min\{k | u^i > w^k > u\}.$$

(Note that $j > i$.) Each u^{i+1} is well-defined since otherwise u^i would be a successor of u. It is clear that $\{u^i\}$ is strictly decreasing. To prove that $u = \inf_i u^i$, let w^n be any monomial larger than u, then since $u^{n+1} = w^j$ for some $j > n$, this implies that $u^n \leq w^n$ by definition of u^{n+1}. ∎

The topology on the ring of formal power series $R = \mathbf{R}[[x]]$ is that of term-wise convergence. This implies that a sum of terms $\sum_i t_i$ converges if $|t_i| \to \infty$ as $i \to \infty$. Here $|\cdot|$ is the total degree: $|cx^\alpha| := \alpha_1 + \cdots + \alpha_n$. Given a map $\Psi : M \to R$, we extend the total degree function to M by setting $|m| := |\widetilde{\Psi} m|$. Now we can state the result:

Proposition 6.39. *Let $\Psi : M \to R$ be a linear map, such that the associated monomial map $\widetilde{\Psi}$ is well-defined. Assume further that M has the topology of term-wise convergence, and that Ψ is continuous. Then Ψ has the normal form property.*

Remark 6.40. By the assumption that $\widetilde{\Psi}$ is well-defined we mean that $\Psi m \neq 0$ for monomials m. Continuity of Ψ is equivalent to the statement that for any monomial $m \in R$ there are only finitely many monomials m' such that $\mathsf{LM}\, \Psi m' = m$.

For the proof we need two more results.

Lemma 6.41. *Let u^i be a strictly decreasing sequence of monomials. Then $|u^i| \to \infty$ as $i \to \infty$.*

Proof: Assume it does not, then there is an infinite, strictly decreasing subsequence $\hat{u}^i$ with $|\hat{u}^i|$ =constant, but there are only finitely many monomials with a given total degree. ∎

Theorem 6.42. *(Transfinite induction) Let P be a property of elements of a set T, and let T be well-ordered. Suppose that for all $u \in T$ we have that if $P(u')$ holds for all $u' < u$, then $P(u)$ holds. (In particular, $P(1)$ is true, where 1 is the smallest element of T.) Then $P(u)$ holds for all $u \in T$.*

Proof: [Wae60, p. 17] Suppose the set $S := \{u \in T | P(u) \text{ does not hold}\}$ is non-empty. By well-orderedness it has a smallest element, say u; in other words $P(u')$ holds for every $u' < u$. This implies, by assumption, that $P(u)$ holds, a contradiction. ∎

Proof of Proposition 6.39: Let $f \in R$ be given. Let T be the set of monomials of R, with the element $-\infty$ adjoined, which is supposed to be smaller than any monomial. Let $P(u)$, with $u \in T$, denote the following property:

$$P(u) \Leftrightarrow \begin{cases} \alpha_u \in M \text{ and } r_u \in R \text{ are well-defined,} \\ \Psi\alpha_u + r_u = f, \\ r_u = 0 \text{ or } \mathsf{LM}\, r_u \notin \operatorname{Im} \widetilde{\Psi} \text{ or } \mathsf{LM}\, r_u \leq u. \end{cases}$$

Assume $P(u')$ holds for all $u' > u$. We show that $P(u)$ also holds.

If $u = 1$, the largest element, then set $\alpha_1 := 0$ and $r_1 := f$, and $P(1)$ holds.

Suppose u has a successor $v > u$, then $P(v)$ holds. If $r_v = 0$ or $\mathsf{LM}\, r_v \notin \operatorname{Im} \widetilde{\Psi}$ or $\mathsf{LM}\, r_v \neq v$, then set $r_u := r_v$ and $\alpha_u := \alpha_v$, making $P(u)$ true. If not, then

$\mathsf{LM}\, r_v \in \operatorname{Im}\widetilde{\Psi}$, say $\mathsf{LT}\, r_v = \widetilde{\Psi} t_v$. Define $\alpha_u := \alpha_v + t_v$ and $r_u := r_v - \Psi t_v$, and $P(u)$ holds.

So suppose $u < 1$ has no successor, then by Lemma 6.38 a decreasing sequence u^i of monomials exists, with infimum u. (This is also true if $u = -\infty$.) Applying Lemma 6.41 we find $|u^i| \to \infty$. From the way the r_{u^i} are constructed, using continuity of Ψ, this implies that the sequence r_{u^i} converges. By definition of total degree on M, also $|t_{u^i}| \to \infty$. Since M has the topology of term-wise convergence, this implies convergence of α_{u^i}. Define $r_u := \lim_{i\to\infty} r_{u^i}$ and $\alpha_u := \lim_{i\to\infty} \alpha_{u^i}$. Again using continuity of Ψ we find $\Psi\alpha_u + r_u = \lim_{i\to\infty}(\Psi\alpha_{u^i} + r_{u^i}) = f$. If r_{u^i} does not become constant from some i on, we have either $r_u = 0$ or $\mathsf{LM}\, r_u \leq v$ for all $v > u$. Since u has no successor the latter inequality implies $\mathsf{LM}\, r_u \leq u$. This shows that $(\forall u' > u : P(u')) \Rightarrow P(u)$.

For the application of Theorem 6.42, reverse the ordering of the monomials. Then the order becomes a well-order (the smallest element of any set of monomials $\{t_i\}$ is $\mathsf{LM} \sum t_i$). The conclusion is that $P(u)$ holds for all $u \in T$, in particular for $u = -\infty$. Define $\mathrm{NF}^{\Psi}(f) := (\alpha_{-\infty}, r_{-\infty})$, then $P(-\infty)$ implies that this is a normal form for Ψ. ∎

Example: Standard ideal- and subalgebra-bases The statement and proofs of the theorems on standard ideal bases (Gröbner bases) and standard subideal bases in polynomial rings, remain unchanged in the present setting of formal power series. The term order is such that 1 is the largest monomial. Again the order is taken to be multiplicative, in the sense that $m_1 < m_2$ implies $mm_1 < mm_2$ for any monomial m for which multiplication is defined. Only the normal form algorithm should be replaced by the normal form map, and the word "Gröbner basis" should be replaced by "standard ideal basis". We refer to sections 6.4.1 and 6.4.3 for details.

6.5.3 Truncated formal power series

In actual computations it is not possible to work with objects that require and infinite amount of data for their description. In particular, we cannot do computations with formal power series. In Sect. 6.5 two solutions to this problem were mentioned, namely restricting to rational functions, and truncating. In this section we describe the latter in more detail.

How to truncate Let us suppose that a map $\Psi : M \to R$ has been fixed, as well as a monomial order on R, and a compatible order on M. A natural and general way of truncating is to restrict to some finite vector space generated by monomials. In this section we give a condition on this vector space R' that makes the truncation "nicely behaved".

The philosophy is as follows. Given spaces M' and R', we restrict $\Psi : M \to R$ to a map between finite dimensional vector spaces $\Psi' : M' \to R'$, by setting $\Psi' := \pi_{R'}\Psi|_{M'}$, where $\pi_{R'}$ is the canonical projection to R'. Then we want that the information contained in Ψ is also contained in the restriction of Ψ, insofar as

it pertains to the monomials in M' and R'. In more precise terms, the following seem natural conditions:

(1) $\operatorname{Im}\Psi' = \pi_{R'} \operatorname{Im}\Psi$,
(2) $(\Psi')\widetilde{\ } = (\widetilde{\Psi})'$,
(3) $\operatorname{Im}\widetilde{\Psi}' = \pi_{R'} \operatorname{Im}\widetilde{\Psi}$,
(4) Ψ is a standard map $\Rightarrow \Psi'$ is a standard map.

It is difficult to see what the weakest condition on M' and R' is, given a map Ψ, such that these conditions are fulfilled. However, a natural condition on R' and M' exists that ensures conditions (1) to (4) above, namely

$$R' := \{m | m \text{ a monomial}, m > m'\},$$
$$M' := \{m \in M | m \text{ a monomial}, \widetilde{\Psi} m \in R'\}, \tag{6.16}$$

for some monomial m'. The form of R' implies that the maps LM and $\pi_{R'}$ commute on R. This in turn implies (2) and (4). The form of M' then implies (1) and (3).

In practice the term orders are multiplicative, and a set R' is finite-dimensional and nontrivial only if the term order is a graded term order. For instance, for pure lexicographic orders the set (6.16) either is infinite-dimensional, or does not contain all variables. The vector space M' is of the same form as R', because of the compatibility of the term orders.

There is another way of describing the truncation. In the case that R is a ring, which includes all of our applications, the span of B is the quotient of R by an ideal, because of the multiplicative property of the term order. This ideal is called the *truncation ideal*. Generators of this ideal include m', but usually it is necessary to include more generators. The truncation ideal is never used in this work.

Normal form For finite sets of monomials, a total order is always a well-order. Therefore algorithms 6.13 and 6.14 terminate, which proves that the normal form property holds in the context of truncated power series, and reduced normal forms exist.

7 Computing normalizing transformations

Given a singularity and a versal deformation of it, an arbitrary deformation can be induced from the versal one by a smooth transformation of coordinates and parameters. Kas and Schlessinger [KS72] developed an algorithm to compute this transformation, for deformations of functions. It requires a procedure to solve the infinitesimal stability equation, which is provided by the normal form map NF^{Ψ} *related to a standard ideal basis (see Chap. 6). A similar approach works for the case of deformations of maps, involving left-right tangent spaces instead of ideals.*

7.1 Introduction

In Chap. 5 we quoted a result from Mather that may be summarized as follows:

A deformation is versal if it is transversal.

The condition of transversality is an *infinitesimal* condition. It is expressed in terms of the tangent space to the orbit of the undeformed singularity, under the group of transformations considered. If this tangent space has finite codimension (in the appropriate function space), and moreover the deformation directions span the quotient space, the deformation is called transversal. In contrast, versality is a *local* condition. A deformation is versal if it can be connected to an arbitrary deformation of the same singularity through smooth reparametrizations and coordinate transformations, on a full neighborhood in parameter- and phase-space. This local property immediately implies the infinitesimal condition. Mather's reverse implication however is a deep result.

The algorithm of Kas and Schlessinger may be viewed as a first step in the direction of Mather's result. From the infinitesimal or linearized condition, it gives a method of computing the formal power series solution of the required reparametrization and coordinate transformation. Mather's result implies existence of such a formal solution, but gives no algorithm to compute it. On the other hand, the existence of a formal solution does not imply existence of a smooth solution in any neighborhood, the existence proof of which involves the Mather-Malgrange preparation theorem.

H. Broer, I. Hoveijn, G. Lunter, and G. Vegter: LNM 1806, pp. 133–151, 2003.

The main ingredient in Kas and Schlessinger's algorithm is a procedure to express any (truncated) formal power series into a 'tangent' part, and a 'transversal' part in some fixed finite-dimensional vector space with a dimension equal to the codimension of the singularity's tangent space. For deformations of functions under right-equivalences, this tangent space is an ideal. Given a standard basis of the ideal, the division algorithm is precisely such a procedure. It splits a truncated formal power series in an element of the ideal, the tangent part, plus a unique rest-term playing the role of the 'transversal' part.

In the case of maps, and left-right transformations, the tangent space has a different structure, but essentially the same ideas apply. The structure of the tangent space (for both cases) is described in Chap. 5. The machinery for computing standard bases for ideals and left-right tangent spaces is described in Chap. 6. The current chapter puts these results together and describes in detail how these are used to compute formal reparametrizations and coordinate transformations, up to any desired order.

Section 7.2 deals with the relatively easy case of functions orbiting under the group of right-transformations. Section 7.3 deals with maps under left-right transformations.

Base ring and formal power series Some of the results in this chapter are formulated in terms of the ring $\mathcal{E}_n$ of germs of functions $\mathbb{R}^n \to \mathbb{R}$. Later we also use rings of formal power series $\mathbb{R}[[x]]$. Both are abstractions of the ring used for actual computations, namely truncated formal power series (over a computable field; see also the remarks in Chap. 6). To avoid excessive notation, we shall in this chapter not explicitly use this truncated ring. As long as a graded term ordering is used, all statement for the full power series ring continue to hold in the truncated setting. See also Sect. 6.5.3.

7.2 Deformations of functions

The problem dealt with in Sect. 2.2.6 can be stated as follows. Suppose a deformation $G(x, v)$ of a function $f(x) = G(x, 0)$ is given. Here $x \in \mathbb{R}^n$ can be thought of as phase space variables, and $v \in \mathbb{R}^c$ as (small) parameters. Now consider the following two problems:

1. Produce a universal deformation $F(x, u) : \mathbb{R}^{n+d} \to \mathbb{R}$ of $f(x)$, and
2. Compute a reparametrization $h : \mathbb{R}^q \to \mathbb{R}^d$ and coordinate transformation $\phi : \mathbb{R}^{n+q} \to \mathbb{R}^n$ inducing G from F.

(See Sect. 5.3 for definitions.) One could say that (1) gives a catalog of possible bifurcations, and (2) gives an index into this catalog for the given family G. The first problem amounts to computing a basis for the vector space $\mathcal{E}_n/J(f)$, where $\mathcal{E}_n$ is the ring of germs of functions on $\mathbb{R}^n$, and $J(f)$ is the Jacobian ideal,

$$J(f) = \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle_{\mathcal{E}_n}.$$

The hypothesis that such a (finite) basis for $\mathcal{E}_n/J(f)$ exists is equivalent to requiring that $J(f)$ has finite codimension. Under this hypothesis, the basis elements can be computed by reducing to truncated formal power series. Chapter 6 describes algorithms to compute an analogue of Gröbner bases for ideals in truncated formal power series rings. Using these bases, called *standard bases*, generators of $\mathcal{E}_n/J(f)$ are easily found.

Suppose $t_1(x), \ldots, t_d(x) \in \mathcal{E}_n$ are such generators. Another way of putting this is to say that the equation

$$g(x) = \alpha_1(x)\frac{\partial f}{\partial x_1} + \cdots + \alpha_n(x)\frac{\partial f}{\partial x_n} + \mu_1 t_1(x) + \cdots + \mu_d t_d(x) \tag{7.1}$$

has a solution in $\alpha_i \in \mathcal{E}_n$ and $\mu_i \in \mathbb{R}$, for arbitrary $g \in \mathcal{E}_n$. Equation (7.1) is called the *infinitesimal stability equation*. Problem 2 is reduced, by an algorithm of Kas and Schlessinger [KS72], to solving (7.1) a number of times, in the ring of truncated formal power series. The standard basis of $J(f)$ is again used to compute the α_i and μ_i efficiently.

We now go into a bit more detail.

7.2.1 Finding a universal deformation

We start by recalling a few results and definitions from Chap. 5, in particular Sect. 5.3.1. A deformation F (of f) is called *versal* if any other deformation G (of f) can be induced from it, and *universal* if it has the minimum possible number of deformation parameters. It is called *transversal* if the 'initial speeds' (see [Mar82]) or 'deformation directions' $\{\frac{\partial F}{\partial u_1}|_{u=0}, \ldots, \frac{\partial F}{\partial u_d}|_{u=0}\}$ complement the tangent space $T_f = J(f)$. By theorem 5.5, a deformation F is versal if and only if it is transversal, reducing the problem of finding a universal deformation of f to finding complements of the ideal $J(f)$ as a vector subspace of $\mathcal{E}_n$.

Proposition 7.1. *Let $f \in \mathcal{E}_n$ be a germ of some function, and suppose that $J(f) \supseteq \mathfrak{m}^p$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{E}_n$. Let $<$ be a term order with respect to which $\mathfrak{m}^p$ is a truncation ideal. Let $\{h_1, \ldots, h_k\}$ be a standard basis w.r.t. $<$ of the projection of $J(f)$ in the ring of truncated formal power series. Then the following monomials forms a basis for $\mathcal{E}_n/J(f)$:*

$$\{m : m \notin \langle \mathsf{LM}\, h_1, \ldots, \mathsf{LM}\, h_k \rangle\}.$$

Suppose the monomials are labeled $t_1, \ldots, t_d$, then a universal deformation of f is given by

$$F(x,u) = f(x) + u_1 t_1(x) + \ldots + u_d t_d(x). \tag{7.2}$$

Proof: Because $\{h_1, \ldots, h_k\}$ is a standard basis, the t_i complement $\mathsf{LM}(J(f) + \mathfrak{m}^p)$. Since $\mathcal{E}_n/\mathfrak{m}^p$ is finite dimensional, this implies that $\mathbb{R}\{t_1, \ldots, t_d\} + J(f) + \mathfrak{m}^p = \mathcal{E}_n$, and using $J(f) \supseteq \mathfrak{m}^p$ also $\mathbb{R}\{t_1, \ldots, t_d\} + J(f) = \mathcal{E}_n$, proving the first

statement. Versality of F follows from theorem 5.5. If one from $\{t_1, \ldots, t_d\}$ is removed, the remaining ones do not complement $J(f)$, so F cannot be versal. This proves universality. ∎

In practice the truncation-order p may not be known beforehand. One chooses a large p and checks afterwards whether $\mathfrak{m}^{p-1} \subseteq J(f)$.

7.2.2 The algorithm of Kas and Schlessinger

We now address the question of computing the transformations that induce arbitrary deformations from a universal one. Suppose $F(x,u) : \mathbb{R}^{n+d} \to \mathbb{R}$ and $G(x,v) : \mathbb{R}^{n+c} \to \mathbb{R}$ are deformations of f, and suppose F is (uni)versal. The goal is to find a parameter-dependent coordinate transformation $\phi : \mathbb{R}^{n+q} \to \mathbb{R}^n$, and a reparametrization $h : \mathbb{R}^q \to \mathbb{R}^d$, satisfying

$$\begin{aligned} \phi(x,0) &= x, \\ h(0) &= 0, \\ G(x,v) &= F(\ \phi(x,v), h(v)\ ). \end{aligned} \tag{7.3}$$

Kas and Schlessinger's algorithm [KS72] accomplishes this, in the ring of truncated formal power series. The idea is to expand ϕ and h with respect to the total degree of the *parameters* v, and solve iteratively for increasing order in v. The details are as follows. Define

$$\phi(x,v) := \sum_{i \geq 0} \phi_i(x,v), \qquad h(v) := \sum_{i \geq 0} h_i(v),$$

where ϕ_i and $h_i(v)$ are homogeneous of degree i in v, and denote the partial sums up to total order p in v by superscripting with the degree:

$$\phi^p(x,v) := \sum_{i=0}^{p} \phi_i(x,v), \qquad h^p(v) := \sum_{i=0}^{p} h_i(v).$$

Now assume that (7.3) has been solved up to order p, that is,

$$G(x,v) = F(\phi^p(x,v), h^p(v)) + O(v^{p+1}) + O(x^m) \tag{7.4}$$

(where it is supposed that we truncate at order m in x). This equation holds for $p = 0$ if we define $\phi^0(x,v) = x$ and $h^0(v) = 0$, since by assumption $F(x,0) = G(x,0) = f(x)$. To solve (7.3) up to order $p+1$ we add $(p+1)$st order terms in v:

$$\begin{aligned} &F(\phi^{p+1}, h^{p+1}) = \\ &F(\phi^p + \phi_{p+1}, h^p + h_{p+1}) = \\ &F(\phi^p, h^p) + D_x F(\phi^p, h^p) \cdot \phi_{p+1} + D_u F(\phi^p, h^p) \cdot h_{p+1} + O\left((\phi_{p+1} + h_{p+1})^2\right) = \\ &F(\phi^p, h^p) + D_x f(\phi^p, h^p) \cdot \phi_{p+1} + D_v F(\phi^p, h^p)|_{v=0} \cdot h_{p+1} + O(v^{p+2}). \end{aligned}$$

To obtain the last equality we used the estimates $\phi^p(x,v) = x+O(v)$, $\phi_{p+1}(x,v) = O(v^{p+1})$, $h^p(v) = O(v)$ and $G(x,v) = f(x) + O(v)$. Now we plug in the explicit form (7.2) for F. Writing $h_{p+1,k}$ and $\phi_{p+1,k}$ for the k-th components of the $p+1$-st order terms of h and ϕ, the equation to be solved for (7.3) to hold up to order $p+1$ is

$$G(x,v) - F\left(\ \phi^p(x,v), h^p(v)\ \right) = \sum_{k=1}^{n} \phi_{p+1,k}(x,v)\frac{\partial f}{\partial x_k} \ + \ \sum_{k=1}^{d} h_{p+1,k}(v)t_k(x) \ + \ O(v^{p+2}). \tag{7.5}$$

By (7.4), the left-hand-side does not contain terms of order less than $p+1$. Equation (7.5) can be solved by equating coefficients of terms v^σ left and right, where $|\sigma| = \sigma_1 + \cdots + \sigma_d = p+1$. For each term v^σ we obtain an equation of the form (7.1), the infinitesimal stability equation. Since $F(x,v)$ is supposed to be a universal deformation, these equations can be solved. This proves existence of a formal power series for ϕ and h (or a solution up to any desired order in x and v) solving (7.3).

The algorithm of Kas and Schlessinger is a recipe for computing ϕ and h, given a procedure for solving (7.1). In practice it is important to be able to solve (7.1) efficiently, as for every term v^σ one such equation is encountered. A procedure that meets this criterion is described in the next section.

Remark 7.2. *(Parameters vs. phase variables)* It is important to note that the transformations obtained converge as power series *in the parameters* v, and not necessarily in the phase variables x. Even if a power series solution in x exists (i.e., no zeroth-order terms are needed), then Kas and Schlessinger's algorithm may not find it, because of the non-uniqueness of the solutions $\alpha_i(x)$ in (7.1).

7.2.3 Solving the infinitesimal stability equation

The problem of computing the transformations inducing a given deformation from a versal one is now reduced to solving (7.5), i.e., (7.1), a number of times. We re-state the problem slightly using the notation of Chap. 6. Write $h_i := \frac{\partial f}{\partial x_i}$, suppose $\langle h_1, \ldots, h_n \rangle$ has finite codimension, and let $t_1, \ldots, t_d$ denote the monomials not in $\mathsf{LM}\langle h_1, \ldots, h_n \rangle$. Then $\langle h_i \rangle + \mathbb{R}\{t_1, \ldots, t_d\} = \mathcal{E}_n$, in other words, for any $g \in \mathcal{E}_n$ it is possible to solve

$$g = \sum_{i=1}^{n} \alpha_i h_i + \sum_{i=1}^{d} \mu_i t_i \tag{7.6}$$

for $\alpha_i \in \mathcal{E}_n$ and $\mu_i \in \mathbb{R}$.

The procedure is as follows. First add new elements $h_i \in \langle h_1, \ldots, h_n \rangle$ $(i > n)$ to the ideal basis, using Buchberger's algorithm (see Sect. 6.2.4), until the basis $\{h_1, \ldots, h_n, h_{n+1}, \ldots, h_k\}$ is a standard basis. The output r of the *reduced*

normal form algorithm 6.14 of Sect. 6.3.7 is an element of $\mathbb{R}\{t_1, \ldots, t_d\}$, that is, can be written as $r = \sum_{i=1}^{d} \mu_i t_i$. Then, the following equation holds:

$$g = \sum_{i=1}^{k} \alpha_i h_i + \sum_{i=1}^{d} \mu_i t_i,$$

where α_i are also output of the reduced normal form algorithm. This equation is, apart from the upper limit in the first sum, of the form (7.6). So if only the numbers μ_i are required, this solves our problem.

If it is also required to express the ideal member $g - r$ in terms of the $h_1, \ldots, h_n$, we can use the extended Buchberger algorithm. To write down this algorithm, we need some notation:

Definition 7.3.
a) $M_k := \bigoplus_{i=1}^{k} \mathcal{E}_n e_i$, *the free k-dimensional* $\mathcal{E}_n$*-module,*
b) $\Psi_k : M_k \to \mathcal{E}_n : fe_i \mapsto fh_i \quad (i = 1, \ldots, k)$, *an* $\mathcal{E}_n$*-module homomorphism,*
c) $\Theta_k : M_k \to M_n$, *an* $\mathcal{E}_n$*-module homomorphism.*

Here the e_i are just symbols, and the h_i are elements of $\mathcal{E}_n$. The algorithm below also uses the notation s_{ij} for the element of M_k (with $k \geq i, j$) whose Ψ-image is the S-polynomial of h_i and h_j. Note that $\mathrm{NF}_{\alpha}^{\Psi_k}$ maps $\mathcal{E}_n$ into M_k. With these notations, the algorithm is the following:

Algorithm 7.4. *(Extended Buchberger algorithm)*
Input: $h_1, \ldots, h_n \in \mathcal{E}_n$
Output: $h_1, \ldots, h_k \in \mathcal{E}_n$ *and an* $\mathcal{E}_n$*-module homomorphism* $\Theta : M_k \to M_n$, *with the properties:*

1. $\{h_1, \ldots, h_k\}$ *is a standard basis for* $\langle h_1, \ldots, h_n \rangle$.
2. $\Psi_k \alpha = \Psi_n \Theta \alpha$, *for any* $\alpha \in M_k$.

Algorithm:

$k \leftarrow n$
$\Theta_k \leftarrow$ *identity map*
While $\mathrm{NF}_{r}^{\Psi_k}(\Psi_k s_{ij}) \neq 0$ *for any* $1 \leq i < j \leq k$, *do:*
  $h_{k+1} \leftarrow \mathrm{NF}_{r}^{\Psi_k}(\Psi_k s_{ij})$
  $\Theta_{k+1}|_{M_k} \leftarrow \Theta_k$, *and*
  $\Theta_{k+1} e_{k+1} := \Theta_k \left(s_{ij} - \mathrm{NF}_{\alpha}^{\Psi_k}(\Psi_k s_{ij}) \right)$
  $k \leftarrow k + 1$
EndWhile
$\Theta \leftarrow \Theta_k$

In an implementation, the map Θ is easily represented by an $n \times k$ matrix of elements of $\mathcal{E}_n$.

Proof: For the proof of termination, and of the fact that $\{h_1, \ldots, h_k\}$ forms a standard basis, see the proof of algorithm 6.17 in Sect. 6.4.1. To prove that

$\Psi_k\alpha = \Psi_n\Theta_k\alpha$ for all k, we proceed by induction. The assertion is trivial for $k = n$, so assume it holds for some k. The definition of Θ_{k+1} assures that it maps into M_n. By hypothesis, $\Psi_{k+1}|_{M_k} = \Psi_n\Theta_{k+1}|_{M_k}$, so it remains to show that $\Psi_n\Theta_{k+1}e_{k+1} = \Psi_{k+1}e_{k+1}$:

$$\begin{aligned}(7.7)\quad \Psi_n\Theta_{k+1}e_{k+1} &= \Psi_n\Theta_k\left(s_{ij} - \mathrm{NF}_\alpha^{\Psi_k}(\Psi_k s_{ij})\right) = \\ &\Psi_k\left(s_{ij} - \mathrm{NF}_\alpha^{\Psi_k}(\Psi_k s_{ij})\right) = \\ &\Psi_k s_{ij} - \Psi_k\,\mathrm{NF}_\alpha^{\Psi_k}(\Psi_k s_{ij}) = \\ &\mathrm{NF}_r^{\Psi_k}(\Psi_k s_{ij}) = h_{k+1} = \Psi_{k+1}e_{k+1},\end{aligned}$$

where in the first step the definition of Θ_{k+1}, and in the second step the induction hypothesis was used. ∎

With the map Θ, solving the infinitesimal stability equation (7.6) becomes trivial: Given $g \in \mathcal{E}_n$, compute $\alpha := \Theta\,\mathrm{NF}_\alpha^{\Psi}(g)$ and $\mu := \mathrm{NF}_r^{\Psi}(g)$, then write them in the form

$$\alpha = \alpha_1 e_1 + \cdots + \alpha_n e_n, \qquad \mu = \mu_1 t_1 + \cdots + \mu_d t_d,$$

with $\alpha_i \in \mathcal{E}_n$ and $\mu_i \in \mathbb{R}$. Now $\alpha_1, \ldots, \alpha_n, \mu_1, \ldots, \mu_d$ solve (7.6).

7.2.4 Application: The hyperbolic umbilic

Here we apply the method of the previous section to the singular germ $x(x^2 + y^2)$. This singularity is commonly known as the hyperbolic umbilic, or D_4^+ in Arnol'd's classification [Arn81]. For this singularity, the general method above is like using a mirror to look at your bangles.[1] However, the underlying idea is the same as in the case of left-right tangent spaces, dealt with in Sect. 7.3, but without the technical complications, so it also serves as an introduction to the next section.

Let $f = x(x^2 + y^2)$, then the generators of $J(f)$ are $h_1 := \frac{\partial f}{\partial x} = 3x^2 + y^2$ and $h_2 := \frac{\partial f}{\partial y} = 2xy$. Take a graded term order with $y < x$, then the only syzygy is $s_{12} = 2ye_1 - 3xe_2$, with $\Psi s_{12} = 2y^3$, which is a monomial not in $\langle \mathsf{LM}\, h_1, \mathsf{LM}\, h_2\rangle$. Therefore $\mathrm{NF}_\alpha^{\Psi_2}(\Psi s_{12}) = 0$ and Θ_3 becomes

$$\Theta_3 e_i = e_i \quad (i = 1, 2) \quad \text{and} \quad \Theta_3 e_3 = 2ye_1 - 3xe_2.$$

New syzygies are $s_{13} = 2y^3e_1 - 3x^2e_2$ and $s_{23} = y^2e_2 - xe_3$, and these are reduced to zero, proving that

$$\{3x^2 + y^2, 2xy, 2y^3\}$$

is a standard ideal basis for $J(f)$. Using algorithm 6.3.7 and the map Θ the infinitesimal stability equation can be solved efficiently.

[1] Hindi proverb

7.3 Deformations of maps

This section is the analogue of Sect. 7.2 for the case of deformations of the energy–momentum map, instead of the planar Hamiltonian function. Let $F(x,u):\mathbb{R}^{n+d}\to\mathbb{R}^2$ be a deformation of $\boldsymbol{E}(x):=F(x,0)$. Universality of these deformations is defined with respect to the class of left-right transformations

$$(A,B):\qquad \boldsymbol{E}\mapsto B\circ\boldsymbol{E}\circ A,$$

where $A:\mathbb{R}^n\to\mathbb{R}^n$ and $B:\mathbb{R}^2\to\mathbb{R}^2$. Suppose that $G(x,v):\mathbb{R}^{n+c}\to\mathbb{R}^2$ is any deformation of $\boldsymbol{E}$. Again there are the two basic problems: Finding a universal deformation F of $\boldsymbol{E}$, and computing a coordinate transformation connecting it to G. Their solution is also similar in spirit to the function-case. For functions, the tangent space was the Jacobian ideal $J(f)$. For maps, the algebraic structure of the tangent space is much more involved, and this results in technical complications. The machinery of Chap. 6 helps us out, however.

The energy–momentum map $\boldsymbol{E}:x\mapsto(H,H_2)$ which we are interested in, is of a special form. The first component is the Hamiltonian function, the second component is its quadratic part. Exploiting this, we arrived in Sect. 3.2.3 at the reduced tangent space:

$$T^r_{\boldsymbol{E}}=J+\{1,f_1,f_2\}\mathbb{R}[[H,H_2]].$$

Here the f_i are functions related to H and H_2, and J is an ideal whose generators are given by equation (3.9). The ability to write any (truncated) formal power series as an element of $T^r_{\boldsymbol{E}}$ plus a rest-term in some finite dimensional vector space, is crucial to computing the coordinate transformation.

7.3.1 Adaptation of Kas and Schlessinger's algorithm

In this section we show how to compute the transformations inducing an arbitrary deformation from a universal one. So suppose $F(x,u):\mathbb{R}^{n+d}\to\mathbb{R}^2$ is a universal deformation of $\boldsymbol{E}$, and let $G:\mathbb{R}^{n+q}\to\mathbb{R}^2$ be an arbitrary deformation of the same map. We are looking for a left-right transformation (A,B), depending on parameters $v\in\mathbb{R}^q$, and a reparametrization $h:\mathbb{R}^q\to\mathbb{R}^d$, such that

$$\begin{aligned}
A(x,0)&=x,\\
B(y,0)&=y,\\
h(0)&=0,\\
G(x,v)&=B\Big(\ F(\ A(x,v),h(v)\ ),v\ \Big)\ =\ B\circ F(\cdot,h)\circ A.
\end{aligned}\tag{7.8}$$

We proceed in the same way as in Kas and Schlessinger's algorithm. Expand A, B and h with respect to the total order in the parameters v, and write

$$A(x,v) := \sum_{i\geq 0} A_i(x,v), \quad B(y,v) := \sum_{i\geq 0} B_i(y,v), \quad h(v) := \sum_{i\geq 0} h_i(v),$$

where A_i, B_i and h_i are homogeneous of degree i in v. Denote the partial sums up to and including order p by superscripting with p:

$$A^p(x,v) := \sum_{i=0}^{p} A_i(x,v), \quad B^p(y,v) := \sum_{i=0}^{p} B_i(y,v), \quad h^p(v) := \sum_{i=0}^{p} h_i(v).$$

Assume that A^p, B^p and h^p solve (7.8) up to order p. This is true for $p = 0$ if we set $A^0(x,v) = x$, $B^0(y,v) = y$ and $h^0(v) = 0$, since $F(x,0) = G(x,0) = \boldsymbol{E}(x)$. To solve (7.8) for the next order $p+1$ we add $(p+1)$st order terms:

$$\begin{aligned} B^{p+1} \circ F(\cdot, h^{p+1}) \circ A^{p+1} &= (B^p + B_{p+1}) \circ F(\cdot, h^p + h_{p+1}) \circ (A^p + A_{p+1}) \\ &= B^p \circ F(\cdot, h^p) \circ A^p \\ &\quad + B_{p+1}(F(A^0(x,v), h^0(v)), v) \\ &\quad + B^0(D_u F(A^0(x,v), u)|_{u=0} \cdot h_{p+1}) \\ &\quad + B^0(D_x F(A^0(x,v), h^0(v))) \cdot A_{p+1}) \;+\; O(|v|^{p+2}) \\ &= B^p \circ F(\cdot, h^p) \circ A^p \\ &\quad + B_{p+1}(\boldsymbol{E}(x), v) \\ &\quad + D_u F(x,u)|_{u=0} \cdot h_{p+1} \\ &\quad + D_x \boldsymbol{E}(x) \cdot A_{p+1} \;+\; O(|v|^{p+2}). \end{aligned}$$

This expression should be equal to $G(x,v)$ up to (but not including) order $p+2$ in v. Since the term $B^p \circ F(\cdot, h^p) \circ A^p$ is already equal to $G(x,v)$ up to $O(|v|^{p+1})$ terms, the other three terms should account for the remaining terms in v of order $p+1$. The resulting equation is

$$\begin{aligned} &(7.9) \quad G(x,v) \;-\; B^p \circ F(\cdot, h^p) \circ A^p \\ &= \quad B_{p+1}(\boldsymbol{E}(x), v) \;+\; \sum_{i=1}^{d} (t_i(x), 0) \cdot h_{p+1,i}(v) \;+\; \boldsymbol{\alpha}_{p+1}(x,v)\boldsymbol{E}(x) \;+\; O(|v|^{p+2}). \end{aligned}$$

Here we wrote $\boldsymbol{\alpha}_{p+1}$ for a vector field on $\mathbb{R}^n$ of order $p+1$ in v, and we used that $F(x,u)$ is of the form

$$F(x,u) = \boldsymbol{E}(x) + u_1(t_1(x), 0) + \cdots + u_d(t_d(x), 0).$$

Note that (7.9) is an equation of *maps* to $\mathbb{R}^2$.

From this point on, we use two facts that are particular to the application to the energy–momentum map. The first is that the projection of the tangent space $T_{\boldsymbol{E}}$ to its second component is surjective, so that deformation terms can be chosen of the form $(t_i, 0)$ (see Sect. 3.2.3). The second fact is more of a condition: We do not want to leave the circle-symmetric setting, in which we found ourselves after the Birkhoff procedure; therefore the vector field $\boldsymbol{\alpha}$ is required to be circle-equivariant.

Collecting terms of the form v^σ of order $p+1$, solving (7.9) boils down to solving, given $\boldsymbol{g}(x)$ and $\boldsymbol{E}(x)$ and $t_i(x)$, several instances of an equation of the form

$$\boldsymbol{g}(x) = \beta(\boldsymbol{E}(x)) + \boldsymbol{\alpha}(x)\boldsymbol{E}(x) + \sum_{i=1}^{d}(t_i(x),0)\mu_i \tag{7.10}$$

for maps $\beta : \mathbb{R}^2 \to \mathbb{R}^2$, circle-equivariant vector fields $\boldsymbol{\alpha}$, and real numbers μ_i. From the assumption that $F(x,u) = \boldsymbol{E}(x)+\sum_i(t_i,0)u_i$ is a universal deformation of $\boldsymbol{E}(x)$ it follows that this equation has a solution.

In Sect. 3.2.3, the tangent space $T_{\boldsymbol{E}}$ is reduced to

$$T^r_{\boldsymbol{E}} = J + \{1, f_1, f_2\}\mathbb{R}[[H, H_2]],$$

where $J = \langle h_0, h_1, h_2\rangle$. With the results of Sect. 6.4.5 we can find functions $t_1, \ldots, t_d$ complementing $T^r_{\boldsymbol{E}}$. More precisely, given an element $g \in \mathbb{R}[[x]]$, there are $a_0, a_1, a_2 \in \mathbb{R}[[x]]$ and $\gamma_0, \gamma_1, \gamma_2 \in \mathbb{R}[[y_1, y_2]]$ and $\mu_1, \ldots, \mu_d \in \mathbb{R}$ such that

$$g(x) = \sum_{i=0}^{2} a_i h_i + \gamma_0(H, H_2) + f_1\gamma_1(H, H_2) + f_2\gamma_2(H, H_2) + \sum_{i=0}^{d}\mu_i t_i. \tag{7.11}$$

We shall refer to equation (7.11) as the *infinitesimal stability equation*, since it is the equivalent of (7.1) in the current context. Solving this equation is the subject of section 7.3.2 below. A solution to (7.11) translates, reading Sect. 3.2.3 backwards, into a solution for (7.10). This works as follows. For the notation, see Sect. 3.2.3.

Suppose the left-hand-side of (7.10) has the form

$$\boldsymbol{g}(x) = (\ g_1(x), g_2(x)\ ).$$

Note that although at the first pass we have $g_2(x) \equiv 0$, since the second component of the unfolding $G(x,v)$ is constant, this may not be true at subsequent passes. Write

$$g_2(\rho_1, \rho_2, \psi) = a_0 + \rho_1 a_1(\rho_1, \rho_2, \psi) + \rho_2 a_2(\rho_1, \rho_2, \psi) + \psi a_3(\rho_1, \rho_2, \psi), \tag{7.12}$$

where $a_0 \in \mathbb{R}$. Using (3.4) or (3.5), we can find a vector field $\boldsymbol{\alpha}'$ such that

$$\boldsymbol{\alpha}' H_2 = g_2 - a_0. \tag{7.13}$$

In fact,

$$\boldsymbol{\alpha}' := \frac{a_1}{q}\left(\rho_1\frac{\partial}{\partial\rho_1} + \frac{1}{2}p\psi\frac{\partial}{\partial\psi}\right) + \frac{a_2}{p}\left(\rho_2\frac{\partial}{\partial\rho_2} + \frac{1}{2}q\psi\frac{\partial}{\partial\psi}\right) + a_3\left(\frac{1}{2q}\psi\frac{\partial}{\partial\rho_1} + \frac{1}{2p}\psi\frac{\partial}{\partial\rho_2} + \rho_1^{p-1}\rho_2^{q-1}\left(\frac{p}{4q}\rho_2 + \frac{q}{4p}\rho_1\right)\frac{\partial}{\partial\psi}\right)$$

works. Now define

$$g := g_1 - \boldsymbol{\alpha}'' H$$

and suppose a_i, γ_i and μ_i solve (7.11) for this g. Recall that the ideal generators h_i of J correspond to generators $\boldsymbol{w}_i$ of the module of circle-equivariant vector fields that leave H_2 invariant:

$$\begin{aligned}
\boldsymbol{w}_1 &= 2\rho_1\rho_2\left(p\frac{\partial}{\partial\rho_1} - q\frac{\partial}{\partial\rho_2}\right) + (p^2\rho_2 - q^2\rho_1)\psi\frac{\partial}{\partial\psi},\\
\boldsymbol{w}_2 &= 2\psi\left(p\frac{\partial}{\partial\rho_1} - q\frac{\partial}{\partial\rho_2}\right) + (p^2\rho_1^{p-1}\rho_2^q - q^2\rho_1^p\rho_2^{q-1})\frac{\partial}{\partial\psi}.
\end{aligned}$$

The generators of J are all defined in terms of the $\boldsymbol{w}_i$ by

$$h_1 := \boldsymbol{w}_1 H, \qquad h_2 := \boldsymbol{w}_2 H,$$

with the exception of h_0, which is the relation. Also recall the definition of $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ (see Sect. 3.2.3). These vector fields obey $\boldsymbol{\alpha}_1 H_2 = H$ and $\boldsymbol{\alpha}_2 H_2 = H_2$, moreover $\boldsymbol{\alpha}_1 H = -f_1$ and $\boldsymbol{\alpha}_2 H = -f_2$. Using this, and the solution of (7.11) we define

$$\begin{aligned}
\beta_1(y_1, y_2) &:= \gamma_0(y_1, y_2),\\
\beta_2(y_1, y_2) &:= y_1\gamma_1(y_1, y_2) + y_2\gamma_2(y_1, y_2) + a_0,\\
\boldsymbol{\alpha} &:= a_1\boldsymbol{w}_1 + a_2\boldsymbol{w}_2 + \boldsymbol{\alpha}' - \gamma_1(H, H_2)\boldsymbol{\alpha}_1 - \gamma_2(H, H_2)\boldsymbol{\alpha}_2.
\end{aligned}$$

We then get

$$\boldsymbol{\alpha}H_2 + \beta_2(H, H_2) \;=\; a_1\boldsymbol{w}_1 H_2 + a_2\boldsymbol{w}_1 H_2 + \boldsymbol{\alpha}' H_2 + a_0 \;=\; g_2$$

(modulo h_0) since $\boldsymbol{w}_i H_2 = 0$ by definition of J, and

$$\begin{aligned}
&\boldsymbol{\alpha}H + \beta_1(H, H_2) \;=\\
&\quad a_1\boldsymbol{w}_1 H + a_2\boldsymbol{w}_2 H + \boldsymbol{\alpha}' H - \gamma_1(H, H_2)\boldsymbol{\alpha}_1 H - \gamma_2(H, H_2)\boldsymbol{\alpha}_2 H + \gamma_0(H, H_2) \;=\\
&\quad a_1 h_i + a_2 h_2 + (g_1 - g) + \gamma_1(H, H_2)f_1 + \gamma_2(H, H_2)f_2 + \gamma_0(H, H_2) \;=\\
&\quad g_1 - \sum \mu_i t_i(x)
\end{aligned}$$

(also modulo h_0), where we used (7.11). These calculations show that $\boldsymbol{\alpha}$ and $\beta = (\beta_1, \beta_2)$ and $\mu_1, \ldots, \mu_d$ solve (7.10). Summarizing, we get the following:

Proposition 7.5. *Suppose $\boldsymbol{E}(x) = (H(x), H_2(x))$ and $\boldsymbol{g}(x) = (g_1(x), g_2(x))$. Write g_2 as in (7.12) and let $\boldsymbol{\alpha}'$ be such that (7.13) holds. Let $g := g_1 - \boldsymbol{\alpha}' H$, and suppose that $a_i \in \mathbb{R}[[x]]$, $\gamma_i \in \mathbb{R}[[y]]$ and $\mu_i \in \mathbb{R}$ are such that (7.11) holds for this g. Then*

$$\begin{aligned}
\boldsymbol{\alpha} &:= a_1\boldsymbol{w}_1 + a_2\boldsymbol{w}_2 + \boldsymbol{\alpha}' - \gamma_1(H, H_2)\boldsymbol{\alpha}_1 - \gamma_2(H, H_2)\boldsymbol{\alpha}_2\\
\beta(y_1, y_2) &:= (\ \gamma_0(y_1, y_2)\ ,\ y_1\gamma_1(y_1, y_2) + y_2\gamma_2(y_1, y_2) + a_0\ )
\end{aligned}$$

together with the μ_i solve (7.10) modulo h_0.

7.3.2 Solving the infinitesimal stability equation

The main building block of the algorithm for computing a left-right transformation inducing the given deformation $G(x, v)$ from the universal deformation $F(x, v)$, is the solution to the *infinitesimal stability equation* (7.11), as was shown in the previous section. If the generators $(\{h_i\}, \{f_i\}, \{g_i\})$ described in Proposition 3.3 form a standard basis of the reduced tangent space

$$T^r_{\boldsymbol{E}} = \langle h_0, h_1, h_2 \rangle + \{1, f_1, f_2\}\mathbb{R}[[g_1, g_2]],$$

the deformation directions t_i are easily found, and the normal form algorithm immediately gives a solution to (7.11).

However, these generators do *not* form a standard basis, and we should add generators to turn it into a one. This is done with a suitable generalization of Buchberger's algorithm, already outlined in Sect. 6.4.5. This creates a new problem: The normal form algorithm now returns a solution to the 'extended' infinitesimal stability equation, the equation with the new generators added to it. This solution cannot be used directly to build the left-right transformation. Just as in Sect. 7.2.3 this problem is solved by extending the Buchberger algorithm so that it computes a map Θ that maps expressions using standard basis generators, to those using only the original generators. In this way a solution to the extended infinitesimal stability equation is mapped to a solution to (7.11).

Now some remarks on Buchberger's algorithm for this case. Since $T^r_{\boldsymbol{E}}$ is an $\mathbb{R}[[g_1, g_2]]$-module, to find a standard basis one can compute the normal forms of syzygies described by lemma 6.31, and add nonzero ones to the $\{f_i\}$ until each syzygy is reduced to 0. Though this algorithm is correct, efficient it is not. The reason is that it always adds nonzero syzygies at the 'coarsest' level of an $\mathbb{R}[[g_1, g_2]]$-module, while for example ideal-syzygies, which are reduced *using ideal generators only* to nonzero normal form, may be added to the ideal generators $\{h_i\}$, thereby reducing the codimension of $\operatorname{Im}\widetilde{\Psi}$ at least as much, and possibly far more, than addition to the module generators $\{f_i\}$ would. Also, adding to the subalgebra-generators as much as possible leads to increased efficiency. An outline of the resulting algorithm is as follows:

a) Expand $\{h_i\}$ to a standard ideal basis of $\langle h_i \rangle$, then
b) Expand $\{g_i\}$ to a standard subalgebra basis of $\mathbb{R}[[g_i]]$, then
c) Expand $\{f_i\}$ to make $(\{h_i\}, \{f_i\}, \{g_i\})$ a standard basis for $\langle h_i \rangle + \{f_i\}\mathbb{R}[[g_i]]$.

Remark 7.6. *(Efficiency)* One could add a stage before the last, namely extending $\{g_i\}$ to make $(\{h_i\}, \{g_i\})$ a standard basis for $\langle h_i \rangle + \mathbb{R}[[g_i]]$. This makes sense since $f_0 = 1$ and therefore syzygies for $\langle h_i \rangle + \mathbb{R}[[g_i]]$ are also syzygies for the left-right tangent space. However the practical improvement turns out to be slight in our case.

The map Θ, mapping back to original generators, is built up during the Buchberger algorithm. We now introduce some notation for the intermediate maps.

Note that we start counting from h_0 and f_0, to be compatible with the notation used throughout.

Definition 7.7.
a) $M_k := \oplus_{i=0}^{k} Re_{1i}$
b) $N_{lm} := \oplus_{i=0}^{l} \mathbb{R}[[y_1, \ldots, y_m]]e_{2i}$
c) $\Theta_{klm} : M_k \oplus N_{lm} \to M_2 \oplus N_{22}$

Note that $T^r_{\boldsymbol{E}}$ has 3 ideal generators (including h_0), 3 module generators (including $f_0 = 1$) and 2 subalgebra generators. The maps Ψ_{klm} are defined, in Sect. 6.4.5, in terms of the basis $\langle h_0, \ldots, h_k\rangle + \{f_0, \ldots, f_l\}\mathbb{R}[[y_1, \ldots, y_m]]$, and map from $M_k \oplus N_{lm}$ to R. On M_k these maps are R-module homomorphisms. On N_{lm} they are $\mathbb{R}[[y_1, \ldots, y_m]]$-module homomorphisms, with obvious multiplication in the domain, while on the range $y_i \cdot f := g_i f$. We also introduce the restricted maps $\Psi^h_k : M_k \to R$ and $\Psi^g_m : N_{0m} \to R$, with the superscripts denoting the generators in terms of which they are defined.

With this notation we can write down the extended Buchberger algorithm. Note that in the text below, whenever a new map $\Theta_{(k+1)lm}$ is defined, it is understood to coincide with the previous Θ_{klm} on $M_k \oplus N_{lm}$.

Algorithm 7.8. *(Extended Buchberger algorithm for LR-tangent spaces)*
Input: $h_0, h_1, h_2, f_0 = 1, f_1, f_2, g_1, g_2 \in R$
Output: *Elements* h_i, f_i, g_i *and a map* $\Theta_{klm} : M_k \oplus N_{lm} \to M_2 \oplus N_{22}$ *such that*

a) $(\{h_0, \ldots, h_k\}, \{f_0, \ldots, f_l\}, \{g_1, \ldots, g_m\})$ *is a standard basis for the left-right tangent space* $\langle h_i\rangle + \{f_i\}\mathbb{R}[[g_i]]$,
b) $\Psi_{klm}\alpha = \Psi_{222}\Theta_{klm}\alpha \quad$ *for all* $\alpha \in M_k \oplus N_{lm}$.

Algorithm:

$k \leftarrow 2, \quad l \leftarrow 2, \quad m \leftarrow 2$
$\Theta_{222} \leftarrow$ *Identity map*
While $\mathrm{NF}_r^{\Psi^h_k}(\Psi^h_k(s_{ij})) \neq 0$ *for any* $1 \leq i < j \leq k$, *do:*
$\quad h_{k+1} \leftarrow \mathrm{NF}_r^{\Psi^h_k}(\Psi^h_k(s_{ij}))$
$\quad \Theta_{(k+1)lm}e_{1(k+1)} := \Theta_{klm}\left(s_{ij} - \mathrm{NF}_\alpha^{\Psi^h_k}(\Psi^h_k(s_{ij}))\right)$
$\quad k \leftarrow k + 1$
EndWhile
Compute subalgebra-syzygies $\{b_i\}$ *using Algorithm 6.24*
While $\mathrm{NF}_r^{\Psi^g_m}(\Psi^g_m(b_i)) \neq 0$ *for any* i, *do:*
$\quad g_{m+1} \leftarrow \mathrm{NF}_r^{\Psi^g_m}(\Psi^g_m(b_i))$
$\quad \Theta_{kl(m+1)}y_{m+1} := \Theta_{klm}\left(b_i - \mathrm{NF}_\alpha^{\Psi^g_m}(\Psi^g_m(b_i))\right)$
$\quad m \leftarrow m + 1$
$\quad$ *Re-compute syzygies* $\{b_i\}$ *for basis* $\{g_1, \ldots, g_m\}$
EndWhile
Compute LR-tangent space syzygies (Lemma 6.31,6.15; Alg. 6.26, 6.34, 6.36)

While $\mathrm{NF}_r^{\Psi_{klm}}(\Psi_{klm}(S)) \neq 0$ *for syzygy* S *of 2nd or 3rd kind, do:*
$f_{l+1} \leftarrow \mathrm{NF}_r^{\Psi_{klm}}(\Psi_{klm}(S))$
$\Theta_{k(l+1)m} e_{2(l+1)} := \Theta_{klm}\left(S - \mathrm{NF}_\alpha^{\Psi_{klm}}(\Psi_{klm}(S))\right)$
$l \leftarrow l+1$
Re-compute syzygies for basis $(\{h_1 \dots h_k\}, \{f_1 \dots f_l\}, \{g_1 \dots g_m\})$
EndWhile

Proof: In general the output $\mathrm{NF}_r^{\Psi}(f)$ is equal to f modulo $\mathrm{Im}\,\Psi$. Since $f \in \mathrm{Im}\,\Psi$ it follows that $h_{k+1} \in \langle h_1, \dots, h_k \rangle$, $g_{m+1} \in \mathbb{R}[[g_1, \dots, g_m]]$ and $f_{l+1} \in \{h_i\} + \{f_i\}\mathbb{R}[[g_i]]$. This with the fact that the condition in the final While-loop is false at exit proves part (a), since syzygies of the first kind reduce to zero as $\{h_1, \dots, h_k\}$ has been extended to a standard ideal basis in the first part of the algorithm.

For part (b) we show that $\Psi_{klm}\alpha = \Psi_{222}\Theta_{klm}\alpha$ is an invariant of the algorithm. Indeed, for $k = l = m = 2$ equality is trivial, and for other values invariance follows, by induction, from the equality $\mathrm{NF}_r^{\Psi}(\Psi S) = \Psi(S - \mathrm{NF}_\alpha(\Psi S))$, which holds for any S and any map Ψ for which a normal form is defined. This proves part (b).

Termination is guaranteed if we work inside the finite dimensional vector space of truncated formal power series, since each new generator increases the set $\langle \mathsf{LM}\, h_i \rangle + \{\mathsf{LM}\, f_i\}\mathbb{R}[[\mathsf{LM}\, g_i]]$ by at least one monomial, by definition of the normal form. ∎

7.3.3 Example of a LR-tangent space calculation

This section is intended to show how algorithm 7.8 works on an easy example. The example was chosen so as to exhibit the essential features of a generic problem instance, but has no extra 'meaning' by itself. We consider the following LR-tangent space:

$$T = \left\langle xy^2, x^2y + a_1y^4 + a_2x^4 \right\rangle + \{1, xy\}\mathbb{R}[[a_3(x^2+y^2), a_4x^4 - a^5x^5]]. \tag{7.14}$$

This is the image of $\Psi : M_2 \oplus N_{22} \to R$ as follows:

$$\begin{aligned} M_2 &= \{e_{11}, e_{12}\}R \\ N_{22} &= \{e_{21}, e_{22}\}\mathbb{R}[[y_1, y_2]] \\ \Psi \;:\; & e_{11}r \mapsto h_1 r \qquad (r \in \mathbb{R}) \\ & e_{12}r \mapsto h_2 r \\ & e_{21}Y \mapsto f_1 Y(g_1, g_2) \qquad (Y \in \mathbb{R}[[z_1, z_2]]) \\ & e_{22}Y \mapsto f_2 Y(g_1, g_2). \end{aligned}$$

Below we shall also use the restricted maps $\Psi^h : M_2 \to R$ and $\Psi^g : \mathbb{R}[[z_i]] \to R$.

The variables $a_1, \dots, a_5$ are coefficients, and are treated as constants. During the calculation certain nondegeneracy-conditions in terms of these coefficients are encountered. These can be read off from the final results.

Standard ideal basis Following algorithm 7.8, first we find a standard ideal basis for $\langle xy^2, x^2y + a_1y^4 + a_2x^4 \rangle$. As a term order we choose the graded order with $x < y$. The first (and only) syzygy between the generator's leading monomials is

$$s_{12} = xe_{11} - ye_{12}$$

whose image under Ψ^h is $-a_2yx^4 - a_1y^5$. This element reduces to $-a_1y^5 + a_2a_1x^2y^4 + a_2^2x^6$ by addition of a_2x^2 times the second generator. The result has leading monomial y^5, which is not in $\operatorname{Im}\widetilde{\Psi}^h = \langle xy^2, x^2y \rangle$. Adding it to the basis (with pre-image e_{13}) gives rise to two new nontrivial syzygies. The syzygy

$$s_{13} = -a_1y^3e_{11} - xe_{13}$$

maps to $-a_2^2x^7 - a_2a_1x^3y^4$, which needs no reduction since its leading monomial is x^7. The other syzygy, nor those involving x^7, give new elements. The final standard ideal basis is $\{h_1, h_2, h_3, h_4\} :=$

$$\{xy^2\ ,\ x^2y + a_1y^4 + a_2x^4\ ,\ -a_1y^5 + a_2a_1x^2y^4 + a_2^2x^6\ ,\ -a_2^2x^7 - a_2a_1x^3y^4\}.$$

Using the expressions for the syzygies, and the subsequent reduction, the new basis elements h_3, h_4 can be expressed in the original ones. Algorithm 7.8 collects this information in the map Θ. As a map from M_4 to M_2, it takes the following form:

$$\Theta = \begin{pmatrix} 1 & 0 & x & -x^2 - a_1y^3 \\ 0 & 1 & -y + a_2x^2 & xy - a_2x^3 \end{pmatrix}.$$

For example, from the fourth column it follows that h_4 is equal to $(-x^2 - a_1y^3)h_1 + (xy - a_2x^3)h_2$. Figure 7.1 depicts the standard basis, and the leading monomial occurring in the ideal.

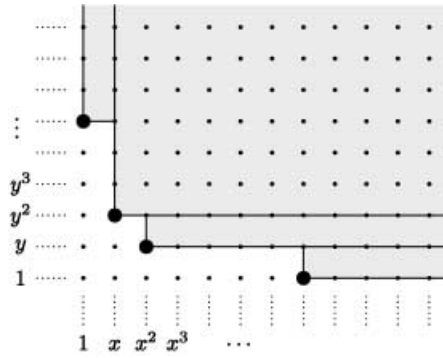

Fig. 7.1 Leading monomials of an ideal, and of its standard basis.

Standard subalgebra basis In the second part of the algorithm, the set $\{g_1, g_2\} = \{a_3(x^2+y^2), a_4x^4 - a_5y^4\}$ is extended to a standard subalgebra basis. The syzygies involved are called the *syzygies of the first kind*, see Chap. 6. These syzygies may be computed using algorithm 6.24. It begins by computing a Gröbner basis for the ideal

$$\left\langle z_1 - a_3x^2, z_2 - a_4x^4 \right\rangle$$

with respect to the elimination term order with $\{z_1, z_2\} < \{x, y\}$. The resulting Gröbner basis is

$$\{a_4z_1^2 - a_3^2z_2, z_1 - a_3x^2\}.$$

Here z_1 and z_2 correspond to the respective generators of the subalgebra. Following algorithm 6.24 we select those basis elements that do not depend on x or y. There is only one such generator, yielding the syzygy $a_4z_1^2 - a_3^2z_2$. Its image under Ψ^g is $2a_3^2a_4x^2y^2 + a_3^2(a_5+a_4)y^4$, and since its leading monomial is x^2y^2 this element cannot be reduced. A second round gives a larger Gröbner basis, but the only binomial involving only the z_i is the one already considered. The standard subalgebra basis thus becomes $\{g_1, g_2, g_3\} :=$

$$\{a_3(x^2+y^2), \quad a_4x^4 - a_5y^4, \quad 2a_3^2a_4x^2y^2 + a_3^2(a_5+a_4)y^4\}.$$

This completes the second stage. The leading monomials of elements of the subalgebra are shown in Fig. 7.2. Those associated with generators are shown as thick bullets.

The data in Fig. 7.2 leaves open the possibility that g_2, with leading monomial x^4, is now superfluous, since there is another generator with leading monomial x^2. Indeed g_2 can be written in terms of the other two, and may be deleted from the generating set, resulting in what could be called a *minimal standard basis*. Such a reduction may shorten subsequent calculations, however we shall not use this.

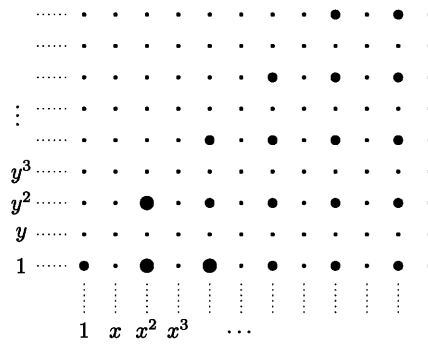

Fig. 7.2 Leading monomials of a subalgebra, and of its standard basis.

Standard left-right tangent space basis Now for the third and final stage. This stage deals with syzygies of the *second* and *third kind*. These involve at least one element of the form $f_i Y(g_1, \ldots, g_m)$. If the other element is also of this form, but with a different f_i, we speak of *syzygies of the third kind*. (The same f_i leads to subalgebra-syzygies, or syzygies of the first kind). If the other element is from the ideal $\langle h_i \rangle$, it is a *syzygy of the second kind* – see Chap. 6 for more details.

The syzygies are found by a Gröbner basis calculation. To implement this stage efficiently, all syzygies found are reduced by the normal form algorithm, with nonzero ones being added to the basis, before the syzygies of the extended basis are re-calculated.

The resulting basis may contain superfluous elements. However the reduction of the calculation time is significant, while a larger basis amounts to only a slight overhead for the normal form calculations. The superfluous elements may also be identified and deleted after the standard basis calculation, if desired. The resulting basis may again be called a *minimal standard basis*, in analogy with Gröbner bases (see e.g. [CLO92, p. 90]).

Let's first calculate the syzygies of the third kind, between elements of the ideal and the subalgebra-module. In the notation of lemma 6.29, we need generators $z^\beta e_{2i}$ of the N_{32}-submodule of elements that map into $\mathsf{LM}\langle h_1, h_2, h_3 \rangle$ under $\widetilde{\Psi}_2 = \widetilde{\Psi}_{N_{32}}$. This is what algorithm 6.34 calculates.

At this point the two generators of the algebra-module are $f_0 = 1$ and $f_1 = x^2y^2$. The generator $t_2 - f_1$ makes the variable t_2 play the role of f_1. Of course f_1^2 and higher powers are in general not in T, but we will filter out these higher powers of t_2 later, which is possible by choice of term order. By adding the generators $z_i - g_i$, $i = 1, 2, 3$, we let the variables z_1, z_2, z_3 play the role of subalgebra-generators g_1, g_2, g_3. Finally, the monomial generators of $\mathsf{LM}\langle h_1, h_2, h_3 \rangle$ are just $\mathsf{LM}\, h_1, \mathsf{LM}\, h_2, \mathsf{LM}\, h_3$ themselves, since they already formed a standard ideal basis. So we have the generators

$$\{t_2 - x^2y^2, z_1 - a_3x^2, z_2 - a_4x^4, z_3 - 2a_3^2a_4x^2y^2, xy^2, x^2y, -a_1y^5, -a_2^2x^7\},$$

a Gröbner basis with respect to the elimination term order with $\{z_i\} \preccurlyeq \{t_2\} \preccurlyeq \{x, y\}$ of which is

$$\{z_3, a_4z_1^2 - a_3^2z_2, z_2^2, t_2z_1, t_2z_2, t_2^2, xz_1z_2, xt_2, yz_1, yz_2, yt_2a_3x^2 - z_1, xy - t_2, y^5\}.$$

We are after the monomials involving only the z_i, and t_2 in degree at most 1. Using algorithm 6.36 we find the following generating set:

$$\{z_3, t_2z_1, t_2z_2, z_2^2\}.$$

These monomials correspond to the elements g_3, f_1g_1, f_1g_2 and g_2^2 with leading monomials x^2y^2, x^3y, x^5y and x^8 respectively, indeed lying in $\mathsf{LM}\langle h_i \rangle$. This gives only one term of the binomial syzygy. The other term is computed by subtracting a multiple of some ideal generator, such that the leading monomial vanishes. The

first syzygy is then e.g. $g_3 - 2a_3^2 a_4 x h_1$, which has leading monomial y^4 and cannot be reduced. This gives a new subalgebra-module generator, $f_2 = a_3^2(a_4 + a_5)y^4$. The second syzygy is $f_1 g_1 - a_3 x h_2$ with leading monomial xy^3. This one can be reduced, via h_1. The result has leading monomial x^5 and cannot be reduced further, giving a second new subalgebra-module generator $f_3 = -a_3 x(a_2 x^4 + a_1 y^4)$. The other two elements can be reduced to 0.

Next, we compute the syzygies of the second kind, among elements of the subalgebra-module. The algorithm computing these is described in Proposition 6.26, and leads to two syzygies:

$$a_4 g_1^2 - a_3^2 g_2 \qquad \text{and} \qquad -a_4 a_3^2(a_4 + a_5)g_3 - 4a_3^4 a_4^2 f_2 g_2.$$

Since after the second stage the algebra generators form a standard basis, the first syzygy, which does not involve the module generators, is guaranteed to reduce to zero. The second must be checked, and turns out also to reduce to zero.

To complete the third stage we have to check all syzygies of the third kind involving the new generators f_2 and f_3. There are four new syzygies:

$$\begin{array}{ll} f_2 g_1 - a_2^2 a_3(a_4 + a_5)xy^2 h_1, & f_3 g_1 - \frac{a_3^2 x^7}{a_2} h_4, \\ f_2 g_2 - a_2^2 a_4(a_4 + a_5)x^3 y^2 h_1, & f_3 g_2 - \frac{a_3 a_4}{a_2} x^2 h_4, \end{array}$$

which all reduce to 0. The final result is in Fig. 7.3, with the monomials associated to the f_i shown as bullets. Summarizing, we found the following module generators in the final stage:

$$f_1 = xy, \qquad f_2 = a_3^2(a_4 + a_5)y^4, \qquad f_3 = -a_3(a_2 x^5 + a_1 xy^4).$$

Codimension The monomials *not* in the tangent space T span a complement. In this case a complement is formed by the set

$$\{x, x^3, y, y^2, y^3\},$$

and the codimension of the LR-tangent space is 5.

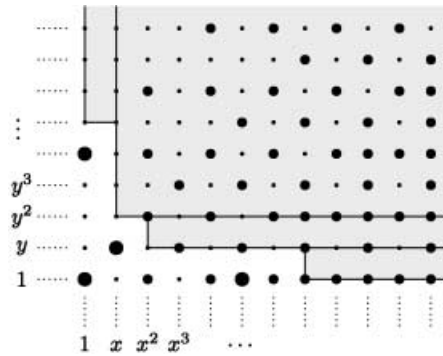

Fig. 7.3 Leading monomials in a LR-tangent space.

Nondegeneracy conditions The set $\mathsf{LM}\,T$ of leading monomials looks like Fig. 7.3 only if the coefficients associated to those monomials are nonzero. In this case, there are four nondegeneracy-conditions. The following table lists them, together with the related generators, and some monomials that will disappear from $\mathsf{LM}\,T$ if the condition is not met:

Condition:	Generators:	Monomials:
$a_1 \neq 0$	h_3	y^5
$a_2 \neq 0$	h_4, f_3	x^7, x^5
$a_3 \neq 0$	g_1	x^2
$a_4 + a_5 \neq 0$	f_2	y^4

Note that the leading coefficient $2a_3^2a_4$ of g_3 does not contribute a nondegeneracy-condition, since the associated monomial x^2y^2 (and multiples) is already contained in the ideal-part of $\mathsf{LM}\,T$.

Appendix

A.1 Classification of term orders

A term order $<$ for Gröbner bases is required to have the following properties:

1. $m \geq 1$, (Noetherian)
2. If $m' \leq m''$ then $mm' \leq mm''$, (Multiplicative)
3. The order $<$ is a total order.

(For the definition of total order, see Sect. 6.3.2.) Such orders are called *admissible orders*. For term orders in formal power series rings, property 1, Noetherianness, is not required. Here we shall use the term *admissible order* to refer to term orders satisfying property 2 and 3. It turns out to be possible to classify all admissible term orders, and to describe them in a uniform way.

In order to formulate the result, we need the following ordering of univariate polynomials. For polynomials $f, g \in \mathbb{R}[Z]$, define $f \geq g$ if and only if $\mathsf{LC}(f-g) \geq 0$, where the ordinary term ordering for univariate polynomials ($1 < Z < Z^2 < \cdots$) is supposed. This is equivalent to the perhaps more intuitive definition

$$f \geq g \qquad \Leftrightarrow \qquad f = g \;\text{ or }\; f(x) - g(x) \to \infty \quad (x \to \infty).$$

Let $\eta_i \in \mathbb{R}[Z]$ for $i = 1, \ldots, n$ be n rationally independent polynomials, then a term order on the variables $x_1, \ldots, x_n$, or equivalently on vectors in $\mathbb{N}^n$, can be defined as follows:

Definition A.1. *(Term order $<_\eta$) For $\alpha, \beta \in \mathbb{N}^n$, let $\alpha <_\eta \beta$ if and only if $\sum_i \alpha_i \eta_i < \sum_i \beta_i \eta_i$.*

The result of this section is that this is in fact the most general way of defining term orders:

Proposition A.2. *For every admissible term order $<$ on $x_1, \ldots, x_n$ there exists a vector of univariate polynomials η such that $<$ coincides with $<_\eta$. Moreover, the polynomials have degree at most $n-1$.*

This result is proved in [Rob85] and [Wei87]. Here we give a different proof using nonstandard analysis (see [Rob88] for a nice introduction), which is both shorter and, in our opinion, more intuitive. The idea is as follows. It is easy to show that

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a linear form on vectors in $\mathbb{N}^n$ satisfying some nondegeneracy conditions defines a term order. Conversely, every term order is 'close' to such a order defined by a linear form, in the sense that given any term order and a *finite* set of monomials, there exists a linear form such that the corresponding term order coincides with the given order on the finite set. The actual term order is therefore a 'limit' of such linear form orders, in the appropriate sense. Taking the limit is technically unpleasant, but this part becomes straightforward using nonstandard analysis.

The first part of the proof is based on the proof of [Stu96, Proposition 1.11], for which we need the following familiar lemma:

Lemma A.3 (Farkas). *[Sch86, Ch. 7.3] Let A be a matrix and b a row vector. Then there exists a row vector $y \geq 0$ with $yA = b$ if and only if $bx \geq 0$ for every column vector x with $Ax \geq 0$.*

Here $x \geq 0$ for a vector x means that every component of x is nonnegative.

Proof of proposition A.2: Let $M \in \mathbb{N}$ be unbounded, and let α_i for $i = 1, \ldots, M^n$ be the elements of $[0, M-1]^n \subset \mathbb{N}^n$ ordered in such a way that $i < j \Leftrightarrow \alpha_i < \alpha_j$. Define $\alpha_{ij} := \alpha_j - \alpha_i$, and

$$C_{ij} := \{x \in \mathbb{R}^n | \alpha_{ij} \cdot x \geq 0\},$$
$$C := \bigcap_{1 \leq i < j \leq M^n} C_{ij}.$$

(The set C_{ij} contains all vectors x such that $\alpha_i \leq_x \alpha_j$, where x is interpreted as a vector of polynomials of degree 0.) We claim that $C \neq \{0\}$. Suppose on the contrary that C only contains the zero vector. This means that there is no nonzero vector x such that $\alpha_{ij} \cdot x \geq 0$ for all $i < j$. If A is the matrix consisting of the row vectors α_{ij} with $i < j$, this implies that the condition of Farkas' lemma is trivially satisfied, so there exist nonnegative real numbers y_{ij} such that

$$\sum_{1 \leq i < j \leq M^n} y_{ij} \alpha_{ij} = (-1, -1, \ldots, -1). \tag{A.1}$$

Since the α_{ij} are rational, we may suppose that the y_{ij} are too. Then, by clearing denominators, (A.1) can be written as

$$\sum_{1 \leq i < j \leq M^n} y'_{ij} \alpha_{ij} = -b, \tag{A.2}$$

where the y'_{ij} are nonnegative integers, and b is some nonnegative integer vector.

Note that the multiplicative property and total orderedness together imply the more general

$$m_1 < m_2 \text{ and } m_3 \leq m_4 \quad \Rightarrow \quad m_1 m_3 < m_2 m_4.$$

For all $i < j$ we have $\alpha_i < \alpha_j$, and using the general multiplicative property this implies in particular that $\sum y'_{ij} \alpha_i < \sum y'_{ij} \alpha_j$. On the other hand, $\alpha_{ij} = \alpha_j - \alpha_i$

so (A.2) can be written as $\sum y'_{ij}\alpha_j + b = \sum y'_{ij}\alpha_i$, that is, $\sum y'_{ij}\alpha_j < \sum y'_{ij}\alpha_i$. This contradiction shows that $C \neq \{0\}$.

Now choose a nonzero vector $\omega \in C$, and recursively define the following sequence of vectors:

$$\begin{aligned} \omega^0 &:= \omega, \\ \widetilde{\omega}^i &:= \frac{\omega^i}{\max_{j=1,\ldots,n} |\omega^i_j|}, \\ \widehat{\omega}^i &:= \mathrm{st}(\widetilde{\omega}^i), \\ \omega^{i+1} &:= \widetilde{\omega}^i - \widehat{\omega}^i. \end{aligned}$$

Here $\mathrm{st}(x)$ denotes the standard part of x. Every $\widetilde{\omega}^i$ has at least one coefficient equal to ± 1, and the corresponding coefficient of $\widehat{\omega}^i$ is also ± 1, so ω^{i+1} becomes zero there, and stays zero for increasing i. Hence, the number of zero entries in ω^i is at least i. Now assume that for some smallest $k < n$ we have that ω^{k+1} is the zero vector. We claim that $\eta := \sum_{i=0}^{k} Z^{k-i}\widehat{\omega}^i$ is the required vector of polynomials. Observe that the components of η are polynomials with standard coefficients, of degree at most $n-1$.

We can write $\omega = \sum_{i=0}^{k} \zeta_i \widehat{\omega}^i$, where the ζ_i are positive constants. Note that ζ_{i+1}/ζ_i is infinitesimal, since ω^{i+1} is a vector with infinitesimal coefficients for $i = 0, \ldots, k-1$. Now let α and β be distinct standard integer vectors with $\alpha < \beta$. Because M is unbounded we have that $\alpha, \beta \in [0, M-1]^n$, and this implies that $\alpha <_\omega \beta$, or

$$\sum_{i=0}^{k} \zeta_i \widehat{\omega}^i \cdot (\alpha - \beta) < 0.$$

Now let $0 \leq t \leq k$ be the smallest integer such that $\sum_{i=0}^{t} \zeta_j \widehat{\omega}^j \cdot (\alpha - \beta) \neq 0$. This means that $\widehat{\omega}^i \cdot (\alpha - \beta) = 0$ for $i = 0, \ldots, t-1$, and $\widehat{\omega}^t \cdot (\alpha - \beta) \neq 0$. Dividing by ζ_t, we can write

$$\widehat{\omega}^t \cdot (\alpha - \beta) < -\sum_{i=t+1}^{k} \frac{\zeta_i}{\zeta_t} \widehat{\omega}^i \cdot (\alpha - \beta).$$

The left hand side is standard and nonzero, the right hand side is infinitesimal, so we find $\widehat{\omega}^t \cdot (\alpha - \beta) < 0$. Together with $\widehat{\omega}^i \cdot (\alpha - \beta) = 0$ for $i = 0, \ldots, t-1$ this implies that $\alpha <_\eta \beta$. This proves that $<_\eta$ coincides with $<$ for any pair of standard integer vectors. Since both term orders are standard, by the Transfer axiom of nonstandard analysis they are identical. ∎

A.2 Proof of Proposition 5.8

We need the following version of Nakayama's lemma. For the proof see e.g. [Mar82, Ch. 1], or [Was74, p. 8].

Lemma A.4 (Nakayama). *Let K and L be $\mathcal{E}^\Gamma$-modules, then*

$$K + \mathfrak{m}L \supset L \quad \Rightarrow \quad K \supset L.$$

Here $\mathfrak{m}$ is the unique maximal ideal in $\mathcal{E}^\Gamma$ of germs of functions vanishing at the origin. We also need the following lemma. It is a symmetric version of the fundamental geometric lemma. See [Mar82] for a proof.

Lemma A.5. *(Symmetric geometric lemma) Let $F(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a t-dependent family of Γ-invariant functions, defined on a neighborhood of $(t,x) \in [0,1] \times \{0\}$, and suppose there exists a vector field $X \in \mathbf{V}_n^\Gamma$ of the form*

$$X = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(t,x)\mathbf{v}_i$$

(where X_i are Γ-invariant families of functions and $\mathbf{v}_i$ are generators of $\mathbf{V}_n^\Gamma$ as a module over $\mathcal{E}_n^\Gamma$), defined on a neighborhood $(t,x) \in [0,1] \times \{0\}$, such that $XF = 0$. Then there exists a Γ-equivariant germ of a diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $F(0,\phi(x)) = F(1,x)$ and $\phi(0) = 0$.

Proof (of Proposition 5.8. Parts (a) and (b) are based on [Mar82, IV.4.2]): We first introduce some notation. Let l be the integer such that $M = M_k \oplus \cdots \oplus M_l$. Let π denote the projection $\pi : \mathfrak{m}_k \to M$. Let $\alpha_{im} \in \mathfrak{m}$ be homogeneous germs such that α_{im} is of degree m, and such that the set $\{\pi\alpha_{im}\}_{im}$ forms a basis of M. The generators of $\mathbf{V}_n^\Gamma$ are $\mathbf{v}_i$, in particular $T_f = \langle \mathbf{v}_i(f) \rangle_{\mathcal{E}_n^\Gamma}$.

We write $g = f + h$, where g is the germ that is supposedly isomorphic to f. We have $h \in \mathfrak{m}_k$ by hypothesis.

(a, first part) The first part consists of proving that $T_{f+th} \supset \mathfrak{m}_k$ for $t \in [0,1]$. By hypothesis, $T_f \supset \mathfrak{m}_k$, so we can find λ_{ijm} such that

$$\alpha_{im} = \sum_j \lambda_{ijm}\mathbf{v}_i(f).$$

Next, define the linear operator H on M by

$$H\alpha_{im} := \pi \sum_j \lambda_{ijm}\mathbf{v}_i(h).$$

Using this we find

$$\pi T_{f+th} \supset \operatorname{span}_{\mathbb{R}}\{\sum_j \lambda_{ijm}\mathbf{v}_i(f+th)\}_{im} = \operatorname{span}_{\mathbb{R}}\{(I+tH)\alpha_{im}\}_{im} = \operatorname{span}_{\mathbb{R}}\{\alpha_{im}\}_{im} = M. \tag{A.3}$$

The penultimate equality holds, for $t \in [0,1]$, if $I + tH$ is invertible for these values of t, which is true if $\pi h = \pi(g - f) \in M$ is small enough. Equation (A.3) can also be written as

$$T_{f+th} + \mathfrak{m} \cdot \mathfrak{m}_k \supset \mathfrak{m}_k,$$

and, by Nakayama, this implies $T_{f+th} \supset \mathfrak{m}_k$, proving the first part.

(a, second part) As $h \in \mathfrak{m}_k$, the statement $T_{f+th} \supset \mathfrak{m}_k$ implies that, for any $\tau \in [0,1]$, we can find germs $X_i(t,x) \in \mathcal{E}^{\Gamma}_{1+n}$ defined on some neighborhood of $(t,x) = (\tau, 0)$, so that

$$\sum_i X_i \mathbf{v}_i(f + th) = -h.$$

Now write $F(t,x) = f(x) + th(x)$, and define the vector field $X := \frac{\partial}{\partial t} + \sum_i X_i(t,x)\mathbf{v}_i$, then $XF = 0$. By compactness of $[0,1]$ we can find a finite number of such vector fields that can be combined to one defined on the entire interval. Lemma A.5 now provides the required isomorphism between $F(0,\cdot) = f$ and $F(1,\cdot) = f + h = g$.

(b) The hypothesis $\mathfrak{m} \cdot T_f \supset \mathfrak{m}_k$ implies that there exist $\lambda_{ijm} \in \mathfrak{m}$ such that

$$\alpha_{im} = \sum_j \lambda_{ijm} \mathbf{v}_i(f).$$

As $h \in \mathfrak{m}_k$ we also have $\mathbf{v}_i(h) \in \mathfrak{m}_k$, so $\lambda_{ijk}\mathbf{v}_i(h) \in \mathfrak{m}^{\Gamma}\mathfrak{m}^{\Gamma}_k$. But $T_{f+th} = \langle \alpha_{im} + t\sum_j \lambda_{ijm}\mathbf{v}_i(h)\rangle_{\mathcal{E}^{\Gamma}_n}$, that is, $T_{f+th} + \mathfrak{m} \cdot \mathfrak{m}_k \supset \mathfrak{m}_k$, and by Nakayama this implies $T_{f+th} \supset \mathfrak{m}_k$. The rest of the proof is the same as the second part of (a).

(c) We assume that the $\mathbf{v}_i$ are homogeneous. (If not, note that $\mathbf{V}^{\Gamma}/(\mathfrak{m} \cdot \mathbf{V}^{\Gamma})$ is finite dimensional, and write $\mathbf{v}_i = \sum_j \mathbf{v}_{ij} + \mathbf{v}_{i,\text{rest}}$ where $\mathbf{v}^0_{ij}$ are finitely many homogeneous terms, and $\mathbf{v}_{i,\text{rest}}$ is an element of $\mathfrak{m}\mathbf{V}^{\Gamma}$, so that $\langle \mathbf{v}_{ij}\rangle_{\mathcal{E}^{\Gamma}_n} + \mathfrak{m}\cdot\mathbf{V}^{\Gamma} = \mathbf{V}^{\Gamma}$. Now use Nakayama to conclude that the $\mathbf{v}_{ij}$ generate $\mathbf{V}^{\Gamma}$ over $\mathcal{E}^{\Gamma}_n$, then use these $\mathbf{v}_{ij}$ instead of the $\mathbf{v}_i$.)

Write f_k for the homogeneous kth degree part of f. We will prove the equivalence $f_k \sim_{\Gamma} g$. The same argument with $g = f$ then proves $f_k \sim_{\Gamma} f$, completing the proof.

First we prove that $T_{f_k} \supset \mathfrak{m}_k$. By hypothesis $h := f - f_k \in \mathfrak{m} \cdot \mathfrak{m}_k$, so we can write $h = h_1 h_k$ with $h_i \in \mathfrak{m}_i$. $\mathbf{v}_i$ maps $\mathfrak{m}_j$ into itself, so $\mathbf{v}_i(h) = h_1\mathbf{v}_i(h_k) + \mathbf{v}_i(h_1)h_k \in \mathfrak{m} \cdot \mathfrak{m}_k$, or $\mathbf{v}_i(f) \in T_{f_k} + \mathfrak{m} \cdot \mathfrak{m}_k$. So we have

$$\mathfrak{m}_k \subset T_f = \langle \mathbf{v}_i(f)\rangle_{\mathcal{E}^{\Gamma}_n} \subset T_{f_k} + \mathfrak{m} \cdot \mathfrak{m}_k.$$

Applying Nakayama we find $T_{f_k} \supset \mathfrak{m}_k$. This inclusion implies the existence of λ_{ijm} such that

$$\alpha_{im} = \sum_j \lambda_{ijm} \mathbf{v}_j(f_k),$$

and, as g_{im}, $\mathbf{v}_j$ and f_k are homogeneous, we may assume that the λ_{ijm} are too.

Now write $g = f_k + h_k + h_{>k}$, where h_k is homogeneous of degree k, and $h_{>k}$ only contains terms of degree $k+1$ and higher. We define the operators H_k and $H_{>k}$ on M by

$$H_{(>)k}\alpha_{im} := \pi \sum_j \lambda_{ijm}\mathbf{v}_j h_{(>)k}.$$

We prove that $H_{>k}$ is nilpotent. Let $\deg(f)$ denote the total degree of a homogeneous germ f, $\operatorname{sdeg}(f)$ the smallest total degree of terms of f, and set $\deg(0) = \operatorname{sdeg}(0) = \infty$. Then

$$\operatorname{sdeg}(H_{>k}\alpha_{im}) \geq \min_j(\deg(\lambda_{ijm}) + \operatorname{sdeg}(\mathbf{v}_j(h_{>k}))) > \\ > \min_j(\deg(\lambda_{ijm}) + \deg(\mathbf{v}_j^0(f_k))) = \deg(\alpha_{im}) = m,$$

so $H_{>k}$ maps M_m into $M_{m+1} \oplus M_{m+2} \oplus \cdots \oplus M_l$, so it is nilpotent, say $H_{>k}^{j_0} = 0$. The operator $I + t(H_k + H_{>k})$ is invertible, for $t \in [0,1]$, if H_k is small enough, i.e., if $\pi(f_k - g) = \pi(f - g)$ is small enough. Indeed, the inverse is given by the sum

$$(I + t(H_k + H_{>k}))^{-1} = \sum_{j=0}^{\infty}(-t(H_k + H_{>k}))^j, \tag{A.4}$$

and nilpotency of $H_{>k}$ allows us to derive the inequality $\|(H_k + H_{>k})^j\| \leq C\|H_k^{j-j_0}\|$, where C is some constant, so that for small H_k, (A.4) converges. We have now:

$$\pi T_{f+t(h_k+h_{>k})} \supset \operatorname{span}_{\mathbb{R}}\{\sum_j \lambda_{ijm}\mathbf{v}_i(f + t(h_k + h_{>k}))\} = \\ \operatorname{span}_{\mathbb{R}}\{(I + t(H_k + H_{>k})g_{im}\}_{im} = \operatorname{span}_{\mathbb{R}}\{g_{im}\} = M, \qquad (t \in [0,1])$$

where we used that $I + t(H_k + H_{>k})$ is invertible. Now apply Nakayama to conclude that $T_{f+th} \supset \mathfrak{m}_k$, where $h = h_k + h_{>k}$. The rest of the proof is the same as the second part of (a). ∎

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